Cramér-Rao Lower Bounds of DOA Estimates from Square QAM-Modulated Signals

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***Abstract*—In this paper, we derive for the first time analytical expressions for the inphase/quadrature (I/Q) non-data-aided (NDA) Cramér-Rao lower bounds (I/Q NDA CRLBs) of the direction of arrival (DOA) estimates from square quadrature amplitude (QAM)-modulated signals corrupted by additive white circular complex Gaussian noise (AWCCGN) with any antenna configuration. Yet the main contribution embodied by this paper consists in deriving for the first time analytical expressions for the NDA Fisher information matrix (FIM) and then for the stochastic CRLB of the NDA DOA estimates in the case of square QAM-modulated signals. It will be shown that in the presence of any unknown phase offset (i.e., non-coherent estimation), the ultimate achievable performance on the NDA DOA estimates holds almost the same irrespectively of the modulation order. However, the NDA CRLBs obtained in the absence of the phase offset (i.e., coherent estimation) vary, in the high SNR region, from one modulation order to another.**

***Index Terms*—QAM signals, DOA estimation, stochastic Cramér-Rao lower bound (CRLB), ULA, UCA.**

I. Introduction

In estimation theory, the performance of any unbiased estimator is often assessed by computing and plotting its bias and variance as a function of the true SNR values. A given unbiased estimator is usually said to outperform another, over a given SNR range, if it exhibits lower variance. In this context, a well known common lower bound for the variance of unbiased estimators of an intended parameter is the CRLB. It serves as a useful benchmark for practical estimators [2]. The CRLB is often numerically or empirically computed. But even when a closed-form expression can be obtained, it is usually complex and requires tedious algebraic manipulations.

Contrarily to the deterministic CRLB which is known to be not achievable in the general case, the stochastic CRLB lends itself as a more accurate bound that can be achieved asymptotically (in the number of measurements) by the stochastic maximum likelihood (ML) estimator and hence there has been much interest in developing this bound. In the context of DOA estimation, several works which deal with the computation of the stochastic CRLB have been reported in the literature. In fact, an explicit expression of the DOA CRLBs for real Gaussian distributions was earlier derived in [3, 4] by Slepian and Bangs. This work was later extended to circular complex Gaussian distributions in [5]. The stochastic and deterministic DOA CRLBs were also derived in [6], where both the signal and noise are jointly circular Gaussian for the stochastic model and deterministic and circular Gaussian for the deterministic model, respectively. More recently, an explicit expression for the stochastic DOA CRLB of noncircular Gaussian sources in the general case of an arbitrary unknown Gaussian noise field was derived in [7].

However, despite the very rich literature on the problem of direction finding, there is not so much information about DOA estimation from modulated sources, especially regarding the bounds on estimation accuracy. In this context, we cite the recent work [8] carried out by Delmas and Abeida who, for the first time, successfully addressed this challenging problem, but only for BPSK and QPSK signals. In another work related to [8], the same authors derived the CRLBs of the DOA parameters from BPSK-, QPSK- and MSK-modulated signals corrupted by a nonuniform Gaussian noise [9]. But to the best of our knowledge, no contribution has dealt so far with the stochastic CRLB for higher-order modulated signals in DOA estimation.

In this paper, we derive explicit expressions for the stochastic inphase/quadrature (I/Q) CRLB of the NDA DOA estimates from arbitrary square QAM-modulated signals. These novel results generalize the elegant CRLB expressions for the DOA estimates from both BPSK-and QPSK-modulated signals presented in [8] to higher-order square QAM modulations.

This paper is organized as follows. In Section II, we introduce the system model that will be used throughout the article.In Section III is devoted to the major contribution embodied by this paper, where simple and explicit expressions for both the NDA FIM and the stochastic I/Q NDA CRLB of the DOA estimates from arbitrary square QAM-modulated signals will be derived. Simulation results are presented in Section IV and, finally, some concluding remarks are drawn out in Section V. We mention that, throughout this paper, matrices and vectors are represented by bold upper case and bold lower case letters, respectively. Moreover, the operators $\{.\}^*$, $\Re\{.\}$ and $\Im\{.\}$ return the conjugate, the real, and the imaginary parts of any complex number, respectively.

Paper approved by N. Benvenuto, the Editor for Modulation and Detection of the IEEE Communications Society. Manuscript received January 19, 2010; revised September 29, 2010 and December 17, 2010.

This work was supported by a Canada Research Chair in Wireless Communications, PROMPT Inc., the Cooperative Research and Development Program of NSERC, and Ericsson Canada, and presented in part at IEEE Globecom'09 [1].

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Digital Object Identifier 10.1109/TCOMM.2011.050211.100036A

0090-6778/11$25.00 © 2011 IEEE

II. System Model

Consider a linearly-modulated signal impinging on an arbitrary array of M antennas. Over the observation interval, the channel is supposed to be of a constant real gain coefficient S and assumed to introduce an unknown distortion phase ϕ. We suppose that the received signal is AWCCGN-corrupted with an overall noise power σ^2. Assuming a receiver with ideal sample timing and perfect frequency synchronization, the received signal at the output of the array matched filter can be modeled as a complex signal as follows:

$$\boldsymbol{y}(n) = S\ e^{j\phi}\ \boldsymbol{a}\ x(n) + \boldsymbol{w}(n), \quad n = 1, 2, \ldots, N, \qquad (1)$$

where, at time index n, $x(n)$ is the transmitted symbol and $\boldsymbol{a}$ is the steering vector parameterized by the scalar DOA parameter θ. For any planar antenna-array configuration, the steering vector can be written as[1]:

$$\boldsymbol{a} = \left[e^{j2\pi f_0(\theta)}, e^{j2\pi f_1(\theta)}, \ldots, e^{j2\pi f_{M-1}(\theta)}\right]^T, \qquad (2)$$

where $\{f_i(\theta)\}_{i=0,2,\ldots,(M-1)}$ are transformations of the scalar DOA parameter θ, which vary from one configuration to another. In general, we have $||\boldsymbol{a}||^2 = M$, where $||.||$ returns the second norm of any vector. The independent and identically distributed (iid) transmitted symbols $\{x(n)\}_{n=1,2,\ldots,N}$ are assumed to be independent from the noise components $\{\boldsymbol{w}(n)\}_{n=1,2,\ldots,N}$. The noise components $\{\boldsymbol{w}(n)\}_{n=1,2,\ldots,N}$ are modeled by independent and identically distributed (iid) M-variate zero-mean complex circular Gaussian random vectors with independent real and imaginary parts and $\mathrm{E}\{\boldsymbol{w}(n)\boldsymbol{w}(n)^H\} = \sigma^2 \boldsymbol{I}_M$, where $\boldsymbol{I}_M$ is the $(M \times M)$ identity matrix. N stands for the number of the received samples in the observation window. Moreover, to derive standard CRLBs, the constellation energy is supposed to be normalized to one, i.e., $\mathrm{E}\{|x(n)|^2\} = 1$, where $\mathrm{E}\{.\}$ refers to the expectation of any random variable and $|.|$ returns the module of any complex number.

In this work, all the parameters σ, S, ϕ and θ are assumed deterministic but unknown and they are more conveniently stacked in the following parameter vector:

$$\boldsymbol{\alpha} = [\sigma \quad S \quad \phi \quad \theta]^T, \qquad (3)$$

where the superscript T stands for the transpose operator. Moreover, we define the true SNR of the system as follows:

$$\rho \;=\; \frac{S^2}{\sigma^2}. \qquad (4)$$

III. Derivation of the Stochastic I/Q Non-Data-Aided CRLB for DOA Estimates from Square QAM Transmissions

In this section, we assume that the transmitted symbols are drawn from any L-ary square QAM constellation, i.e., $L = 2^{2p}$ $(p = 1, 2, 3, \ldots)$. The transmitted symbols are assumed independent and identically distributed (iid). We organize the derivations in three subsections. In Subsection III-A, we consider a general antenna-array configuration. In Subsections III-B and III-C, we consider ULAs and UCAs, respectively.

A. General Antenna-Array Configuration

Considering the parameter vector defined in (3), the NDA CRLB of the DOA estimates is explicitly defined as:

$$\mathrm{CRLB}(\theta) = [\boldsymbol{I}_{\mathrm{NDA}}^{-1}(\boldsymbol{\alpha})]_{4,4}, \qquad (5)$$

where $\boldsymbol{I}_{\mathrm{NDA}}(\boldsymbol{\alpha})$ is the NDA Fisher information matrix 3whose entries are defined as follows:

$$[\boldsymbol{I}_{\mathrm{NDA}}(\boldsymbol{\alpha})]_{i,j} = -\,\mathrm{E}\left\{\frac{\partial^2 \ln(P[\boldsymbol{Y};\boldsymbol{\alpha}])}{\partial\alpha_i \partial\alpha_j}\right\}, \quad i,j = 1,2,3,4; \qquad (6)$$

with $\boldsymbol{Y}$ being the matrix that contains all the received samples on all the antenna elements during the observation interval, i.e., $\boldsymbol{Y} = [\boldsymbol{y}(1), \cdots, \boldsymbol{y}(N)]$ and $P[\boldsymbol{Y};\boldsymbol{\alpha}]$ being the probability density function (pdf) of $\boldsymbol{Y}$ parameterized by $\boldsymbol{\alpha}$.

First, we show (see appendix A) that the NDA FIM is block-diagonal structured as follows:

$$\boldsymbol{I}_{\mathrm{NDA}}(\boldsymbol{\alpha}) = \begin{pmatrix} \boldsymbol{I}_1^{\mathrm{NDA}}(\sigma, S) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_2^{\mathrm{NDA}}(\phi,\theta) \end{pmatrix}. \qquad (7)$$

Moreover, since we assume the transmitted symbols to be iid, then $\{\boldsymbol{y}(n)\}_{n=1,2,\ldots,N}$ are i.i.d M-dimensional random variables, and therefore the elements of the matrices $\boldsymbol{I}_1^{\mathrm{NDA}}(\sigma, S)$ and $\boldsymbol{I}_2^{\mathrm{NDA}}(\phi,\theta)$ are given by:

$$[\boldsymbol{I}_1^{\mathrm{NDA}}(\sigma, S)]_{i,l} = \;-N\,\mathrm{E}\left\{\frac{\partial^2 \ln(P[\boldsymbol{y}(n);\boldsymbol{\alpha}])}{\partial\alpha_i \partial\alpha_l}\right\}\;, \; i,l = 1,2, \qquad (8)$$

$$[\boldsymbol{I}_2^{\mathrm{NDA}}(\phi, \theta)]_{i,l} = \;-N\,\mathrm{E}\left\{\frac{\partial^2 \ln(P[\boldsymbol{y}(n);\boldsymbol{\alpha}])}{\partial\alpha_{i+2} \partial\alpha_{l+2}}\right\}\;, \; i,l = 1,2. \qquad (9)$$

Actually, the analytical derivations of the eight elements involved in (8) and (9) require tedious algebraic manipulations. We now detail the major algebraic developments required for the averaging step and then give the final results. In fact, under the aforementioned assumptions and for any L-ary QAM constellation, the NDA probability density function of the received vector $\boldsymbol{y}(n)$ parameterized by the parameter vector $\boldsymbol{\alpha}$, $P[\boldsymbol{y}(n);\boldsymbol{\alpha}]$ is obtained by averaging the DA probability density function with respect to the points of the constellation alphabet, $\mathcal{C} = \{x_1, x_2, \ldots, x_L\}$, to yield the following results:

$$P[\boldsymbol{y}(n);\boldsymbol{\alpha}] \;=\; \frac{1}{L\pi^M\sigma^{2M}} \sum_{l=1}^{L} \exp\left\{-\frac{||\boldsymbol{y}(n) - S\ e^{j\phi}\ x_l\ \boldsymbol{a}||^2}{\sigma^2}\right\}. \qquad (10)$$

Furthermore, it can be seen that (10) can be written as:

$$P[\boldsymbol{y}(n);\boldsymbol{\alpha}] = frac1L\pi^M\sigma^{2M} \exp\left\{-\frac{||\boldsymbol{y}(n)||^2}{\sigma^2}\right\} D_{\boldsymbol{\alpha}}(n), \qquad (11)$$

where $D_{\boldsymbol{\alpha}}(n)$ is given by:

[1] Note that $\{f_i(\theta)\}_{i=0}^{M-1}$ should be fixed by choosing an origin of phase due to the constant shift $\Delta\theta$ corresponding to the delay introduced by the channel as the wave propagates form the source to the antenna array. But this shift can always be included in the unknown phase offset, ϕ, introduced by the channel which is considered as nuisance parameter in the paper.

$$D_{\boldsymbol{\alpha}}(n)=\sum_{x_l\in\mathcal{C}}\exp\left\{-\frac{S^2M|x_l|^2}{\sigma^2}\right\}\exp\left\{\frac{2S\Re\{e^{j\phi}x_l\boldsymbol{y}(n)^H\boldsymbol{a}\}}{\sigma^2}\right\}. \quad (12)$$

In this section, considering only square QAM constellations, i.e., $L=2^{2p}$ ($p=1,2,3\ldots$), we derive analytical expressions for the CRLBs as functions of the true SNR values ρ. In fact, the major advantage offered by the special case of these square constellations is that, as shown hereafter, $D_{\boldsymbol{\alpha}}(n)$ and therefore $P[\boldsymbol{y}(n),\boldsymbol{\alpha}]$ can be factorized, making it possible to obtain analytical expressions, as a function of the true SNR ρ, for the FIM elements given by (8) and (9). Indeed, when $L=2^{2p}$ for any $p\geq 1$, we have $\mathcal{C}=\{\pm(2i-1)d_p\pm j(2k-1)d_p\}_{i,k=1,2,\cdots,2^{p-1}}$ where $j^2=-1$ and $2d_p$ is the inter-symbol distance in the I/Q plane. Note that d_p is computed using the assumption of a normalized-energy constellation. That is:

$$\frac{\sum_{l=1}^{2^{2p}}|x_l|^2}{2^{2p}} = 1, \quad (13)$$

which yields the following result:

$$d_p = \frac{2^{p-1}}{\sqrt{2^p\sum_{k=1}^{2^{p-1}}(2k-1)^2}}. \quad (14)$$

We prove, after some algebraic manipulations (see appendix B) that $D_{\boldsymbol{\alpha}}(n)$ can be factorized as follows:

$$D_{\boldsymbol{\alpha}}(n) = 4F_{\boldsymbol{\alpha}}(U(n))F_{\boldsymbol{\alpha}}(V(n)), \quad (15)$$

where

$$F_{\boldsymbol{\alpha}}(t)=\sum_{i=1}^{2^{p-1}}\exp\left\{-\frac{S^2M(2i-1)^2d_p^2}{\sigma^2}\right\}\times \cosh\left(\frac{(2i-1)d_pSt}{\sigma^2}\right), \quad (16)$$

and the real scalar random variables $U(n)$ and $V(n)$ are defined as:

$$U(n) = 2\Re\{e^{j\phi}\boldsymbol{y}(n)^H\boldsymbol{a}\}, \quad (17)$$
$$V(n) = 2\Im\{e^{j\phi}\boldsymbol{y}(n)^H\boldsymbol{a}\}. \quad (18)$$

Consequently, from (11), $P[\boldsymbol{y}(n);\boldsymbol{\alpha}]$ follows as:

$$P[\boldsymbol{y}(n);\boldsymbol{\alpha}] = \frac{4}{L\pi^M\sigma^{2M}}\exp\left\{-\frac{||\boldsymbol{y}(n)||^2}{\sigma^2}\right\}\times F_{\boldsymbol{\alpha}}(U(n))F_{\boldsymbol{\alpha}}(V(n)), \quad (19)$$

and the log-likelihood function of the received samples reduces simply to:

$$\ln(P[\boldsymbol{y}_n;\boldsymbol{\alpha}]) = \ln\left(\frac{4}{L\pi^M}\right)-2M\ln(\sigma)-\frac{||\boldsymbol{y}(n)||^2}{\sigma^2}+ \ln\left(F_{\boldsymbol{\alpha}}(U(n))\right)+\ln\left(F_{\boldsymbol{\alpha}}(V(n))\right). \quad (20)$$

First, we show in Appendix C that $U(n)$ and $V(n)$ are independent. Furthermore, by defining the complex scalar random variable $\vartheta(n)=U(n)+jV(n)=2\ e^{j\phi}\ \boldsymbol{y}(n)^H\boldsymbol{a}$ whose pdf is $P[\vartheta(n);\boldsymbol{\alpha}]=P[(U(n),V(n));\boldsymbol{\alpha}]$, we have:

$$P[\vartheta(n);\boldsymbol{\alpha}]=P[U(n);\boldsymbol{\alpha}]P[V(n);\boldsymbol{\alpha}]. \quad (21)$$

We prove after tedious algebraic manipulations (see Appendix D) the following result:

$$P[U(n);\boldsymbol{\alpha}]=\frac{1}{2^p\sqrt{\pi M\sigma^2}}\exp\left\{-\frac{U(n)^2}{4M\sigma^2}\right\}F_{\boldsymbol{\alpha}}(U(n)), \quad (22)$$

$$P[V(n);\boldsymbol{\alpha}]=\frac{1}{2^p\sqrt{\pi M\sigma^2}}\exp\left\{-\frac{V(n)^2}{4M\sigma^2}\right\}F_{\boldsymbol{\alpha}}(V(n)), \quad (23)$$

implying that $U(n)$ and $V(n)$ are actually independent and identically distributed (iid) according to (22) and (23), respectively. This interesting property will be used henceforth to derive all the expectations.

Now, as it can be seen from (20), due to the factorization of the received samples pdf in (19), the log-likelihood function involves the sum of two analogous terms: $\ln(F_{\boldsymbol{\alpha}}(U(n)))$ and $\ln(F_{\boldsymbol{\alpha}}(V(n)))$. This reduces the complexity of the derivation of its second partial derivatives of (20) and make it possible to obtain analytical expressions for their expected values, which is generally the most challenging step in deriving NDA CRLBs. Indeed, since $U(n)$ and $V(n)$ are iid, then it follows immediately that for $i,l=1,2,3,4$:

$$\mathrm{E}\left\{\frac{\partial^2\ln\left(F_{\boldsymbol{\alpha}}(U(n))\right)}{\partial\alpha_i\partial\alpha_l}\right\}=\mathrm{E}\left\{\frac{\partial^2\ln\left(F_{\boldsymbol{\alpha}}(V(n))\right)}{\partial\alpha_i\partial\alpha_l}\right\}.$$

In the sequel, as an example, we will detail the derivation of $\mathrm{E}\left\{\frac{\partial^2\ln(P[\boldsymbol{y}(n);\boldsymbol{\alpha}])}{\partial\theta^2}\right\}$ and the other terms, involved in (8) and (9), can then be easily derived following the same derivation lines. In fact, we have:

$$\mathrm{E}\left\{\frac{\partial^2\ln\left(P[\boldsymbol{y}(n);\boldsymbol{\alpha}]\right)}{\partial\theta^2}\right\}=2\,\mathrm{E}\left\{\frac{\partial^2\ln\left(F_{\boldsymbol{\alpha}}\left(U(n)\right)\right)}{\partial\theta^2}\right\},$$
$$2\,\mathrm{E}\left\{\frac{\ddot{F}_{\boldsymbol{\alpha}}\left(U(n),\dot{U}(n),\ddot{U}(n)\right)}{F_{\boldsymbol{\alpha}}\left(U(n)\right)}\right\} -2\,\mathrm{E}\left\{\frac{\dot{F}_{\boldsymbol{\alpha}}\left(U(n),\dot{U}(n)\right)^2}{F_{\boldsymbol{\alpha}}\left(U(n)\right)^2}\right\}, \quad (24)$$

where

$$\dot{F}_{\boldsymbol{\alpha}} = \frac{\partial\ln\left(F_{\boldsymbol{\alpha}}\right)}{\partial\theta}, \quad (25)$$
$$\ddot{F}_{\boldsymbol{\alpha}} = \frac{\partial^2\ln\left(F_{\boldsymbol{\alpha}}\right)}{\partial\theta^2}, \quad (26)$$
$$\dot{U}(n) = \frac{\partial U(n)}{\partial\theta} \quad (27)$$
$$\ddot{U}(n) = \frac{\partial^2 U(n)}{\partial\theta^2}. \quad (28)$$

It can be shown that:

$$\mathrm{E}\left\{\frac{\ddot{F}_{\boldsymbol{\alpha}}\left(U(n),\dot{U}(n),\ddot{U}(n)\right)}{F_{\boldsymbol{\alpha}}\left(U(n)\right)}\right\}=\mathrm{E}\left\{\ddot{U}(n)\delta_{\boldsymbol{\alpha},1}\left(U(n)\right)\right\}+ \mathrm{E}\left\{\dot{U}(n)^2\delta_{\boldsymbol{\alpha},2}\left(U(n)\right)\right\}, \quad (29)$$

$$E\left\{\frac{\dot{F}_{\boldsymbol{\alpha}}\left(U(n),\dot{U}(n)\right)^2}{F_{\boldsymbol{\alpha}}\left(U(n)\right)^2}\right\}=\mathrm{E}\left\{\dot{U}(n)^2\delta_{\boldsymbol{\alpha},1}\left(U(n)\right)^2\right\}, \quad (30)$$

where

$$\delta_{\boldsymbol{\alpha},1}\left(U(n)\right) = \frac{Sd_p}{\sigma^2 F_{\boldsymbol{\alpha}}\left(U(n)\right)} \sum_{i=1}^{2^{p-1}} e^{-\frac{MS^2d_p^2(2i-1)^2}{\sigma^2}} \times (2i-1)\sinh\left(\frac{(2i-1)d_pS}{\sigma^2}U(n)\right), \quad (31)$$

$$\delta_{\boldsymbol{\alpha},2}\left(U(n)\right) = \frac{S^2d_p^2}{\sigma^4 F_{\boldsymbol{\alpha}}\left(U(n)\right)} \sum_{i=1}^{2^{p-1}} e^{-\frac{MS^2d_p^2(2i-1)^2}{\sigma^2}} \times (2i-1) \cosh\left(\frac{(2i-1)d_pS}{\sigma^2}U(n)\right). \quad (32)$$

Therefore, the derivation of (24) requires explicit expressions for (29) and (30). We begin by deriving the first term given by (29). In fact, as shown in appendix E, we have:

$$\ddot{U}(n) = -\frac{||\dot{\boldsymbol{a}}||^2}{M}U(n) + \frac{j}{2M}\left(\boldsymbol{a}^H\ddot{\boldsymbol{a}} - \ddot{\boldsymbol{a}}^H\boldsymbol{a}\right)V(n) + z(n),$$

with $\ddot{\boldsymbol{a}} = \dfrac{\partial^2 \boldsymbol{a}}{\partial\theta^2}$ and:

$$z(n) = e^{j\phi}\boldsymbol{w}(n)^H\ddot{\boldsymbol{a}} + e^{-j\phi}\ddot{\boldsymbol{a}}^H\boldsymbol{w}(n) - \frac{1}{M}\left[e^{j\phi}\boldsymbol{w}(n)^H\boldsymbol{a}\boldsymbol{a}^H\ddot{\boldsymbol{a}} + e^{-j\phi}\boldsymbol{a}^H\boldsymbol{w}(n)\ddot{\boldsymbol{a}}^H\boldsymbol{a}\right].$$

Moreover, since $U(n)$ is independent from $V(n)$ and $z(n)$ (see appendices C and E) with $\mathrm{E}\{V(n)\} = \mathrm{E}\{z(n)\} = 0$, then after some algebraic manipulations (see appendix F for more details), we obtain:

$$\mathrm{E}\left\{\ddot{U}(n)\delta_{\boldsymbol{\alpha},1}\left(U(n)\right)\right\} = -\frac{4||\dot{\boldsymbol{a}}||^2S^2d_p^2}{2^p\sigma^2}A_{2;p}, \quad (33)$$

where

$$A_{m;p} = \sum_{k=1}^{2^{p-1}}(2k-1)^m, \quad \forall m \in \mathbb{N}. \quad (34)$$

Moreover, $\dot{U}(n)$ and $U(n)$ are independent (see appendix C). Therefore, we have:

$$\mathrm{E}\left\{\dot{U}(n)^2\delta_{\boldsymbol{\alpha},2}\left(U(n)\right)\right\} = \mathrm{E}\left\{\dot{U}(n)^2\right\}\mathrm{E}\left\{\delta_{\boldsymbol{\alpha},2}\left(U(n)\right)\right\},$$

and we show that:

$$\mathrm{E}\left\{\dot{U}(n)^2\right\} = 2||\dot{\boldsymbol{a}}||^2\,\sigma^2 + 2S^2\,|\boldsymbol{a}^H\dot{\boldsymbol{a}}|^2,$$

$$\mathrm{E}\left\{\delta_{\boldsymbol{\alpha},2}\left(U(n)\right)\right\} = \frac{2S^2d_p^2}{2^p\sigma^4}A_{2;p}.$$

Hence, we obtain:

$$\mathrm{E}\left\{\dot{U}(n)^2\delta_{\boldsymbol{\alpha},2}\left(U(n)\right)\right\} = \frac{4||\dot{\boldsymbol{a}}||^2S^2d_p^2}{2^p\sigma^2}A_{2;p} + \frac{4|\boldsymbol{a}^H\dot{\boldsymbol{a}}|S^4d_p^2}{2^p\sigma^4}A_{2;p}.$$

Finally, we deduce:

$$\mathrm{E}\left\{\frac{\ddot{F}_{\boldsymbol{\alpha}}\left(U(n),\dot{U}(n),\ddot{U}(n)\right)}{F_{\boldsymbol{\alpha}}\left(U(n)\right)}\right\} = \frac{4|\boldsymbol{a}^H\dot{\boldsymbol{a}}|S^4d_p^2}{2^p\sigma^4}A_{2;p}. \quad (35)$$

Now, we will derive the second term given by (30). Once again, using the independence of $\dot{U}(n)$ and $U(n)$ and changing the variable $\frac{U(n)}{\sqrt{2M}\sigma}$ by t, (30) reduces to:

$$\mathrm{E}\left\{\frac{\dot{F}_{\boldsymbol{\alpha}}\left(U(n),\dot{U}(n)\right)^2}{F_{\boldsymbol{\alpha}}\left(U(n)\right)^2}\right\} = \frac{S^2d_p^2}{2^{p-2}\sigma^4}\left(\sigma^2||\dot{\boldsymbol{a}}||^2 + S^2|\boldsymbol{a}^H\dot{\boldsymbol{a}}|\right)Q_p(\rho), \quad (36)$$

where

$$Q_p(\rho) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{q_p^2(\rho,t)}{h_p(\rho,t)}\;e^{-\frac{t^2}{2}}dt, \quad (37)$$

with

$$q_p(\rho,t) = \sum_{k=1}^{2^{p-1}} e^{-(2k-1)^2d_p^2M\rho} \times (2k-1)\sinh\left((2k-1)d_p\sqrt{2M\rho}\;\;t\right), \quad (38)$$

$$h_p(\rho,t) = \sum_{k=1}^{2^{p-1}} e^{-(2k-1)^2d_p^2M\rho}\cosh\left((2k-1)d_p\sqrt{2M\rho}\;\;t\right). \quad (39)$$

Finally, injecting (35) and (36) in (24), we obtain:

$$\mathrm{E}\left\{\frac{\partial^2\ln\left(P[\boldsymbol{y}(n);\boldsymbol{\alpha}]\right)}{\partial\theta^2}\right\} = -\frac{2\rho}{2^{p-2}}I_{4,4},$$

where

$$I_{4,4} = -2^{p-2}\rho|\boldsymbol{a}^H\dot{\boldsymbol{a}}|^2 + d_p^2\left(||\dot{\boldsymbol{a}}||^2 + \rho|\boldsymbol{a}^H\dot{\boldsymbol{a}}|^2\right)Q_p(\rho).$$

Deriving the remaining seven NDA FIM elements using equivalent algebraic manipulations (see Appendix G for more details), we obtain the following results:

$$\boldsymbol{I}_1^{\mathrm{NDA}}(\sigma,S) = \frac{2NM}{2^{p-2}\sigma^2}\begin{pmatrix} I_{1,1} & \frac{2S}{\sigma}[2^{p-2}-H_p(\rho)] \\ \frac{2S}{\sigma}[2^{p-2}-H_p(\rho)] & F_p(\rho)\end{pmatrix}. \quad (40)$$

$$\boldsymbol{I}_2^{\mathrm{NDA}}(\phi,\theta) = \frac{2N\rho}{2^{p-2}}\begin{pmatrix} I_{3,3} & I_{3,4} \\ I_{3,4} & I_{4,4}\end{pmatrix}, \quad (41)$$

where

$$I_{1,1} = 2^{p-1} - 4\;\rho\left[A_{2;p}d_p^2 - G_p(\rho)\right] - 2\;\rho^2Md_p^4A_{4;p}, \quad (42)$$

$$I_{3,3} = -2^{p-2}M^2\rho + Md_p^2\left(1+M\rho\right)Q_p(\rho), \quad (43)$$

$$I_{3,4} = -j\dot{\boldsymbol{a}}^H\boldsymbol{a}\;\left[2^{p-2}M\rho - d_p^2(1+M\rho)Q_p(\rho)\right]. \quad (44)$$

In (40)-(42), $F_p(\rho)$, $G_p(\rho)$ and $H_p(\rho)$ are given by:

$$F_p(\rho) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{f_p^2(\rho,t)}{h_p(\rho,t)}\;e^{-\frac{t^2}{2}}dt, \quad (45)$$

$$G_p(\rho) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{g_p^2(\rho,t)}{h_p(\rho,t)}\;e^{-\frac{t^2}{2}}dt, \quad (46)$$

$$H_p(\rho) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{f_p(\rho,t)g_p(\rho,t)}{h_p(\rho,t)}\;e^{-\frac{t^2}{2}}dt, \quad (47)$$

where

$$f_p(\rho,t) = \sum_{k=1}^{2^{p-1}} e^{-(2k-1)^2 d_p^2 M\rho} \times \Big[(2k-1)d_p\; t \sinh\Big((2k-1)d_p\sqrt{2M\rho}\;\; t\Big) - (2k-1)^2 d_p^2 \sqrt{2M\rho}\;\; \cosh\Big((2k-1)d_p\sqrt{2M\rho}\;\; t\Big)\Big], \quad (48)$$

$$g_p(\rho,t) = \sum_{k=1}^{2^{p-1}} e^{-(2k-1)^2 d_p^2 M\rho} \times \Big[(2k-1)d_p\; t \sinh\Big((2k-1)d_p\sqrt{2M\rho}\;\; t\Big) - (2k-1)^2 d_p^2 \sqrt{\frac{M\rho}{2}}\;\; \cosh\Big((2k-1)d_p\sqrt{2M\rho}\;\; t\Big)\Big]. \quad (49)$$

Recall that the FIM is block diagonal. Therefore, it can be easily inverted to yield the expression for the NDA DOA CRLB in non-coherent (NCO) estimation (i.e., in the presence of an unknown phase offset), $\mathrm{CRLB}_{\mathrm{NDA}}^{\mathrm{NCO}}$, as follows:

$$\mathrm{CRLB}_{\mathrm{NDA}}^{\mathrm{NCO}}(\theta) = \frac{2^{p-2}\left(Md_p^2(1+M\rho)Q_p(\rho) - 2^{p-2}M^2\rho\right)}{2N\rho \det\left(I_2^{\mathrm{NDA}}(\phi,\theta)\right)} \quad (50)$$

where det(.) returns the determinant of any square matrix. After some algebraic manipulations and using the identity $\dot{\boldsymbol{a}}^H \boldsymbol{a} = -\boldsymbol{a}^H \dot{\boldsymbol{a}}$, we obtain an explicit expression for the $\mathrm{CRLB}_{\mathrm{NDA}}^{\mathrm{NCO}}$ as follows[2]:

$$\mathrm{CRLB}_{\mathrm{NDA}}^{\mathrm{NCO}}(\theta) = \frac{A_{2;p}}{N\gamma\;\rho\; Q_p(\rho)}, \quad (51)$$

where the expressions of $A_{2;p}$ and $Q_p(\rho)$ are given by (34) and (37), respectively, and γ is the "geometrical factor" $2\dot{\boldsymbol{a}}^H \boldsymbol{\Pi}_{\boldsymbol{a}}^{\perp} \dot{\boldsymbol{a}}$ with $\boldsymbol{\Pi}_{\boldsymbol{a}}^{\perp} = \boldsymbol{I}_M - \boldsymbol{a}\boldsymbol{a}^H/M$ and $\dot{\boldsymbol{a}} = \frac{\partial \boldsymbol{a}}{\partial \theta}$. It is then easy to see that γ devolps to:

$$\gamma = 2\left(||\dot{\boldsymbol{a}}||^2 - \frac{|\boldsymbol{a}^H\dot{\boldsymbol{a}}|^2}{M}\right), \quad (52)$$

Recall the expression of the DA CRLB in case of non-coherent estimation, $\mathrm{CRLB}_{\mathrm{DA}}^{\mathrm{NCO}}(\theta)$, that was earlier derived in [8] as follows:

$$\mathrm{CRLB}_{\mathrm{DA}}^{\mathrm{NCO}}(\theta) = \frac{1}{N\gamma\;\rho}. \quad (53)$$

Then we can relate the CRLBs, in the DA and NDA modes, under non-coherent estimation as follows[3]:

$$\mathrm{CRLB}_{\mathrm{NDA}}^{\mathrm{NCO}}(\theta) = \frac{A_{2;p}}{Q_p(\rho)}\mathrm{CRLB}_{\mathrm{DA}}^{\mathrm{NCO}}(\theta), \quad (54)$$

from which we notice that the NDA DOA CRLB is factorized into the DA DOA CRLB and another term, the ratio $\frac{A_{2;p}}{Q_p(\rho)}$, that depends only on the modulation order *a priori* and the SNR ρ, not on the array geometry. This term reflects, in fact, the complete ignorance of the transmitted symbols. We will actually see *a posteriori* in Section V that this term holds almost the same irrespectively of the modulation order. Moreover, this ratio is strictly superior to one at low SNR values and is almost equal to one in the high-SNR region. This is because, in this SNR region, the useful signal is not too much corrupted by the additive noise and consequently the training symbols do not provide much additional information about the DOA estimates.

In coherent (CO) estimation (i.e., absence of the phase offset or assuming that the parameter ϕ is known or perfectly recovered) the NDA DOA CRLB, $\mathrm{CRLB}_{\mathrm{NDA}}^{\mathrm{CO}}$, can be obtained by simply inverting the second diagonal element of $\boldsymbol{I}_2^{\mathrm{NDA}}(\phi,\theta)$, i.e., we have:

$$\mathrm{CRLB}_{\mathrm{NDA}}^{\mathrm{CO}}(\theta) = -\frac{1}{\mathrm{E}\left\{\frac{\partial^2 \ln(P[\boldsymbol{y}(n);\boldsymbol{\alpha}])}{\partial\theta^2}\right\}}. \quad (55)$$

Notice, however, that coherent NDA estimation stems from an idealistic scenario. Indeed, in an NDA scheme, the required parameters are blindly estimated within an inherent phase ambiguity. Hence, the phase offset cannot be recovered *a priori*. Yet some *pseudo-coherent* estimation techniques enable its recovery within phase ambiguities that keep the constellation invariant by rotation [11]. For QAM-modulated transmissions, the phase offset estimate would therefore be nominally of the form $\widehat{\phi} = \phi + \frac{k\pi}{2}$ where $k \in \{0,1,2,3\}$ with equal probability. Assuming this more realistic pseudo-coherent estimation scenario, we consider the transformed received vector $\boldsymbol{y}'(n) = \boldsymbol{y}(n)\; e^{-j\widehat{\phi}} = S\; e^{j\Delta\phi}\; \boldsymbol{a}\; x(n) + \boldsymbol{w}'(n)$ where $\Delta\phi = \phi - \widehat{\phi}$ and $\boldsymbol{w}'(n) = \boldsymbol{w}(n)\; e^{-j\widehat{\phi}}$ for $n = 1,2\ldots,N$ are also iid M-variate zero-mean complex circular Gaussian random vectors with $\mathrm{E}\{\boldsymbol{w}'(n)\boldsymbol{w}'(n)^H\} = \sigma^2\boldsymbol{I}_M$. Hence, we can easily prove that for $k = 0,1,2,3$, we have (56) at the top of the next page, where

$$U_0(n) = 2\,\Re\{e^{j\Delta\phi}\boldsymbol{y}'(n)^H\boldsymbol{a}\},$$
$$V_0(n) = 2\,\Im\{e^{j\Delta\phi}\boldsymbol{y}'(n)^H\boldsymbol{a}\}.$$

Consequently, we have for any $k \in \{0,1,2,3\}$:

$$P[\boldsymbol{y}'(n);\boldsymbol{\alpha}] = \frac{1}{4}\sum_{k=0}^{3} P\left[\boldsymbol{y}'(n)|\Delta\phi = \frac{k\pi}{2};\boldsymbol{\alpha}\right], = P\left[\boldsymbol{y}'(n)|\Delta\phi = k\frac{\pi}{2};\boldsymbol{\alpha}\right]. \quad (57)$$

In pseudo-coherent estimation, the NDA CRLB $\mathrm{CRLB}_{\mathrm{NDA}}^{\mathrm{PCO}}$ is therefore given by:

$$\mathrm{CRLB}_{\mathrm{NDA}}^{\mathrm{PCO}}(\theta) = -\frac{1}{\mathrm{E}\left\{\dfrac{\partial^2 \ln\left(P[\boldsymbol{y}'(n);\boldsymbol{\alpha}]\right)}{\partial\theta^2}\right\}}. \quad (58)$$

We can easily show from (19), (56) and (57) that $P[\boldsymbol{y}'(n);\boldsymbol{\alpha}]$ and $P[\boldsymbol{y}(n);\boldsymbol{\alpha}]$ have the same form. Then, from (55) and (58), replacing $\boldsymbol{y}(n)$ by $\boldsymbol{y}'(n)$ and following the same derivation lines used previously, we establish that the NDA CRLBs in both coherent and pseudo-coherent estimations are the same:

$$\mathrm{CRLB}_{\mathrm{NDA}}^{\mathrm{CO}}(\theta) = \mathrm{CRLB}_{\mathrm{NDA}}^{\mathrm{PCO}}(\theta) = \frac{2^{p-2}}{2N\rho\Psi_\rho(\theta)}, \quad (59)$$

where

$$\Psi_\rho(\theta) = -2^{p-2}\rho|\boldsymbol{a}^H\dot{\boldsymbol{a}}|^2 + d_p^2(||\dot{\boldsymbol{a}}||^2 + \rho|\boldsymbol{a}^H\dot{\boldsymbol{a}}|^2)Q_p(\rho). \quad (60)$$

[2]Note also that replacing p by 1 in (51) yields the same expression recently presented by Delmas and Habti in [8] in the special case of 4-QAM constellation.

[3]See [12] For further results about the relationship between the DA CRLBs under coherent and non-coherent estimations

$$
\begin{aligned}
P\left[\boldsymbol{y}'(n)|\Delta\phi = k\frac{\pi}{2};\boldsymbol{\alpha}\right] &= \frac{1}{L\pi^M\sigma^{2M}} \times \sum_{l=1}^{L} \exp\left\{-\frac{||\boldsymbol{y}'(n) - S\ e^{j\Delta\phi}\ x_l\ \boldsymbol{a}||^2}{\sigma^2}\right\} \\
&= \frac{4}{L\pi^M\sigma^{2M}} \exp\left\{-\frac{||\boldsymbol{y}'(n)||^2}{\sigma^2}\right\} \times F_{\boldsymbol{\alpha}}(U_0(n))F_{\boldsymbol{\alpha}}(V_0(n))
\end{aligned} \tag{56}
$$

We mention here that no obvious relation can be drawn between $\text{CRLB}_{\text{NDA}}^{\text{CO/PCO}}$ in (59) and the DA CRLB in coherent estimation that was earlier derived in [8] as follows:

$$
\text{CRLB}_{\text{DA}}^{\text{CO}}(\theta) = \frac{1}{2\ N\ \rho\ ||\dot{\boldsymbol{a}}||^2}. \tag{61}
$$

Note that, although the expressions of the NDA CRLBs given by (51) and (59) are valid for any antenna configuration, we will subsequently elaborate more on the the expressions of these bounds for the most popular antenna-array configurations, namely ULAs and UCAs. Indeed, by further examining the exact expressions of γ, $||\dot{\boldsymbol{a}}||^2$ and $|\boldsymbol{a}^H\dot{\boldsymbol{a}}|^2$ involved in (51) and (60), some interesting results on the achievable performance will be revealed regarding these special configurations.

B. Uniform Linear Array (ULA) Configuration

For a uniform linear array (ULA), the steering vector is given by:

$$
\boldsymbol{a} = \left[1, e^{j\pi\sin(\theta)}, e^{2j\pi\sin(\theta)}, \ldots, e^{j(M-1)\pi\sin(\theta)}\right]^T.
$$

Consequently, from (52), γ_{ULA} can be written as:

$$
\begin{aligned}
\gamma_{\text{ULA}} &= 2\pi^2\cos^2(\theta)\left(\sum_{k=1}^{M-1}k^2 - \frac{\left(\sum_{k=1}^{M-1}k\right)^2}{M}\right), \\
&= \pi^2\frac{M(M^2-1)}{6}\cos^2(\theta).
\end{aligned} \tag{62}
$$

Therefore, in the special case of a ULA configuration, we obtain the explicit expression as a function of θ for the NDA CRLB in non-coherent estimation as follows:

$$
\text{CRLB}_{\text{NDA}}^{\text{NCO}}(\theta) = \frac{6 \times A_{2;p}}{NM(M^2-1)\pi^2\cos^2(\theta)\rho Q_p(\rho)}. \tag{63}
$$

Moreover, for a ULA configuration, $|\boldsymbol{a}^H\dot{\boldsymbol{a}}|^2$ and $||\dot{\boldsymbol{a}}||^2$ are given explicitly by:

$$
\begin{aligned}
|\boldsymbol{a}^H\dot{\boldsymbol{a}}|^2 &= \pi^2\cos^2(\theta)\left(\sum_{k=1}^{M-1}k\right)^2, \\
&= \pi^2\frac{M^2(M-1)^2}{4}\cos^2(\theta). \\
||\dot{\boldsymbol{a}}||^2 &= \pi^2\frac{M(M-1)(2M-1)}{6}\cos^2(\theta).
\end{aligned}
$$

Consequently, from (59) and (60), we obtain, for a ULA configuration, the following expression for the NDA CRLB with coherent/pseudo-coherent estimation:

$$
\begin{aligned}
\text{CRLB}_{\text{NDA}}^{\text{CO/PCO}}(\theta) = 3 \times 2^{p-1}\Bigg(& NM(M-1)\pi^2\rho\cos^2(\theta) \times \\
&\Big[-3M(M-1)\rho 2^{p-2} + d_p^2\Big(2(2M-1) + \\
&3M(M-1)\rho\Big)Q_p(\rho)\Big]\Bigg)^{-1}.
\end{aligned} \tag{64}
$$

Therefore, it can be seen from (63) and (64) that for both non-coherent and coherent estimations, the CRLB is an even function of θ and minimal for $\theta = 0$ since $\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, thereby reflecting the symmetry of the ULA around the broadside axis.

C. Uniform Circular Array (UCA) Configuration

For a uniform circular array (UCA), the steering vector is given by [10]:

$$
\boldsymbol{a} = \left[e^{\frac{j\pi\cos(\theta)}{2\sin(\pi/M)}}, e^{\frac{j\pi\cos(\theta-2\pi/M)}{2\sin(\pi/M)}}, \ldots, e^{\frac{j\pi\cos(\theta-2(M-1)\pi/M)}{2\sin(\pi/M)}}\right]. \tag{65}
$$

Hence, from (52), γ_{UCA} is expressed as:

$$
\begin{aligned}
\gamma_{\text{UCA}} &= \frac{\pi^2}{2\sin^2\left(\frac{\pi}{M}\right)} \times \\
&\left(\sum_{k=0}^{M-1}\sin^2\left(\theta - \frac{2k\pi}{M}\right) - \frac{\left(\sum_{k=1}^{M-1}\sin\left(\theta-\frac{2k\pi}{M}\right)\right)^2}{M}\right).
\end{aligned}
$$

Using the identities $\sin\left(\theta - \frac{2k\pi}{M}\right) = \Im\left\{e^{j\left(\theta-\frac{2k\pi}{M}\right)}\right\}$, $\cos\left(\theta - \frac{2k\pi}{M}\right) = \Re\left\{e^{j\left(\theta-\frac{2k\pi}{M}\right)}\right\}$ and $\sum_{k=0}^{M-1} e^{\frac{2jk\pi}{M}} = 0$, we show that γ_{UCA} reduces simply to:

$$
\gamma_{\text{UCA}} = \frac{M\pi^2}{4\sin^2\left(\frac{\pi}{M}\right)}, \tag{66}
$$

Moreover, for a UCA configuration, $|\boldsymbol{a}^H\dot{\boldsymbol{a}}|^2$ and $||\dot{\boldsymbol{a}}||^2$ are given, respectively, by:

$$
\begin{aligned}
|\boldsymbol{a}^H\dot{\boldsymbol{a}}|^2 &= \frac{\pi^2}{4\sin^2\left(\frac{\pi}{M}\right)}\left(\sum_{k=1}^{M-1}\sin\left(\theta-\frac{2k\pi}{M}\right)\right)^2 = 0. \\
||\dot{\boldsymbol{a}}||^2 &= \frac{M\pi^2}{8\sin^2\left(\frac{\pi}{M}\right)}, \\
&= \frac{\gamma_{\text{UCA}}}{2}.
\end{aligned}
$$

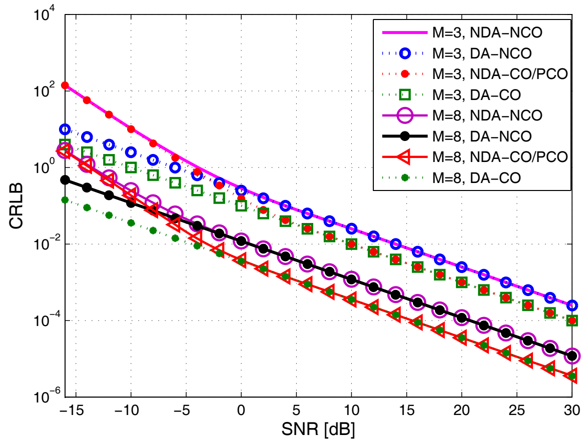

Fig. 1. DA and NDA CRLBs for the DOA estimates, 4-QAM, $N = 1$.

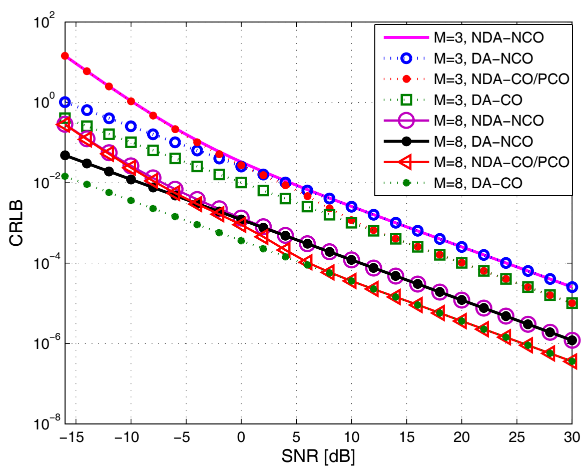

Fig. 2. DA and NDA CRLBs for the DOA estimates, 16-QAM, $N = 1$.

Consequently, from (51), (59) and (60), we obtain, for a UCA configuration, the same expressions for the NDA CRLBs for coherent/pseudo-coherent and noncoherent estimation as follows:

$$\mathrm{CRLB}_{\mathrm{NDA}}^{\mathrm{CO/PCO}}(\theta) = \mathrm{CRLB}_{\mathrm{NDA}}^{\mathrm{NCO}}(\theta) = \frac{4\sin^2\left(\frac{\pi}{M}\right)}{NM\pi^2\rho}\frac{A_{2;p}}{Q_p(\rho)}. \quad (67)$$

Thus, we conclude from (67) that, contrarily to the ULA configuration, the *a priori* knowledge of the phase offset with a UCA configuration does not bring any additional CRLB performance gain. Moreover, we see that the true DOA parameter does not appear in the final CRLB expression implying that the achievable performance for a UCA is the same for any θ, thereby reflecting the circular symmetry of the UCA.

IV. Graphical Representations

In this section, we provide some graphical representations of the NDA and the DA CRLBs of both coherent and noncoherent DOA estimations as a function of the SNR for different modulation orders, different values of the number of antennas M and for a fixed DOA parameter $\theta = 0$. Unless specified otherwise, the receiving antennas are assumed to be distributed as a ULA.

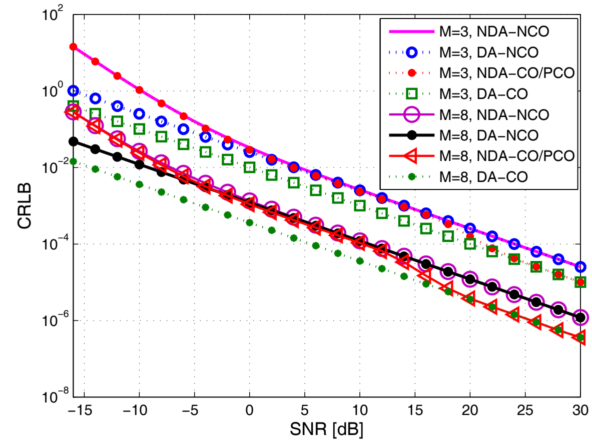

Fig. 3. DA and NDA CRLBs for the DOA estimates, 256-QAM, $N = 1$.

In Fig. 1, we consider the same expression for the steering vector adopted in [8]:

$$\boldsymbol{a} = [1, e^{j\theta}, e^{2j\theta}, \ldots, e^{j(M-1)\theta}]^T,$$

just for comparison purposes. It can be verified there that for $p = 1$ there is a perfect agreement between our analytical expressions and those derived in the special case of QPSK by Delmas and Abeida in [8]. Furthermore, we see from Figs. 1, 2 and 3 that, for relatively high SNR values, the DOA CRLBs obtained in coherent estimation, $\mathrm{CRLB}^{\mathrm{CO/PCO}}$, are lower than those obtained when all the parameters are assumed to be unknown, $\mathrm{CRLB}^{\mathrm{NCO}}$. This is hardly surprising since the more information we exploit, the lower is the bound. In fact, the knowledge of the phase offset is obviously more informative about the unknown DOA parameter at high SNR values. Moreover, we clearly see that the difference between these two CRLBs decreases as the modulation order p increases. We also see from these figures that the non-coherent DA CRLB, $\mathrm{CRLB}_{\mathrm{DA}}^{\mathrm{NCO}}$, of the DOA estimates is lower than the $\mathrm{CRLB}_{\mathrm{NDA}}^{\mathrm{CO/PCO}}$ only for low SNR values when the useful signal is too much corrupted by the additive noise. However, for high SNR values, the phase offset becomes more informative about the DOA parameter than the training symbols.

In addition, it can be seen that the CRLBs decrease when the number of receiving antennas M increases. In fact, the more antennas we have, the more samples we receive and consequently the more information we exploit. Moreover, we notice from these figures that the achievable performance on the DOA estimates holds almost the same irrespectively of the modulation order. This can be analytically explained by examining the explicit expressions of the considered CRLBs in (54). Indeed, the modulation order p (or $L = 2^{2p}$) appears only in the ratio $\frac{A_{2;p}}{Q_p(\rho)}$ which turns out as shown in Fig. 4 to be almost the same for all the considered square QAM constellations (i.e., $p = 1$, 2, 3, 4).

In Fig. 5, we compare the DOA CRLBs for both ULA and UCA antenna-array configurations. We see from this figure that the achievable performance on the DOA estimates

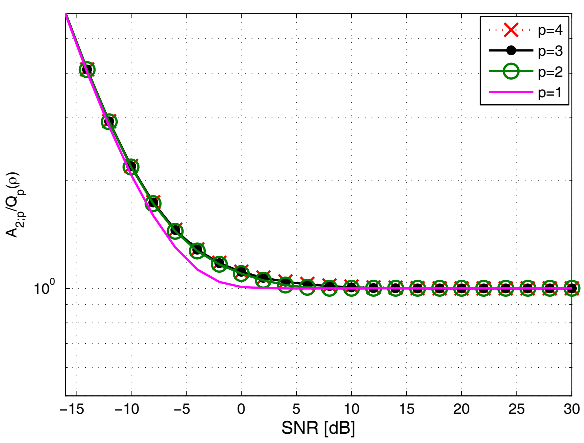

Fig. 4. Ratio $A_{2;p}/Q_p(\rho)$ for $p = 1, 2, 3$ and 4, $M = 8$.

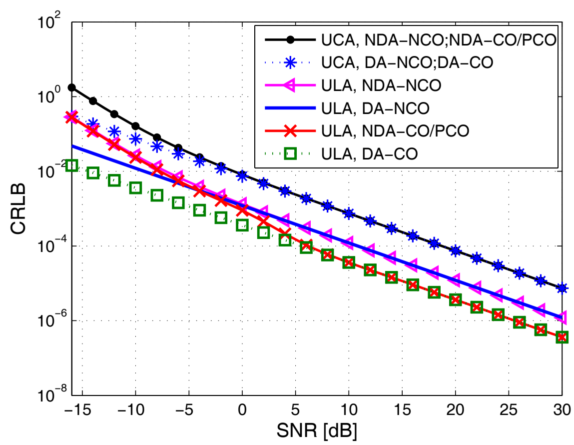

Fig. 5. DA and NDA CRLBs for the DOA estimates with ULA and UCA configurations, $p = 2$, $N = 1$, $M = 8$.

depends on the geometrical configuration of the M receiving antenna elements. In fact, for a fixed number of antennas M, we prove that the CRLBs obtained for a ULA configuration are lower than the CRLBs obtained with a UCA for any value of $\theta \in [-\theta_c, \theta_c]$ with $\theta_c = \arccos\left(\sqrt{\frac{3}{2\sin^2\left(\frac{\pi}{M}\right)(M-1)(M+1)}}\right)$. UCAs are, however, more used in practice. Indeed, the major advantage of UCAs is their 360 degrees azimuthal coverage which is necessary in many applications such as wireless communications, radar and sonar and their almost invariant directional pattern. This is in strong contrast with the widely studied ULA that only covers 180 degrees.

V. Conclusion

In this paper, we developed for the first time analytical expressions for the stochastic CRLBs of the NDA DOA estimates from arbitrary square QAM-modulated signals. We elaborated further on the DA DOA CRLBs for any linearly-modulated signal. The received samples are assumed to be corrupted by additive white circular complex Gaussian noise. We proved that the achievable performance on the NDA DOA estimates in the presence of an unknown phase offset holds almost the same irrespectively of the modulation order and that it depends on the geometrical configuration of the antenna-array. However, the CRLBs obtained in the absence of the phase offset differ from one modulation order to another, in the high SNR region. Moreover, they are lower than the DA CRLBs for non-coherent estimation. In either case, the CRLBs depend in general on the DOA and the antenna-array geometry. With a UCA, we showed that the CRLBs no longer depend on the DOA due to its circular symmetry. With a ULA, we showed that the CRLBs are even functions of the DOA that reach their minima at broadside direction due to the ULA's axial symmetry around it.

Appendix A - Proof of the Diagonal Structure of the NDA FIM

We have for $i = 1, 2$:

$$\mathrm{E}\left\{\frac{\partial^2 \ln\left(P[\boldsymbol{y}(n);\boldsymbol{\alpha}]\right)}{\partial\alpha_i\partial\phi}\right\} = \mathrm{E}\left\{V(n)A_i\left(U(n)\right)\right\},$$

$$\mathrm{E}\left\{\frac{\partial^2 \ln\left(P[\boldsymbol{y}(n);\boldsymbol{\alpha}]\right)}{\partial\alpha_i\partial\theta}\right\} = \mathrm{E}\left\{\dot{U}(n)B_i\left(U(n)\right)\right\},$$

where $\{A_i\}_{i=1,2}$ and $\{B_i\}_{i=1,2}$ are four transformations of the scalar random variable $U(n)$. Moreover, $V(n)$ and $U(n)$ are independent. Therefore, $V(n)$ and any other transformation of $U(n)$ will be independent. Hence, we have:

$$\mathrm{E}\left\{\frac{\partial^2 \ln\left(P[\boldsymbol{y}(n);\boldsymbol{\alpha}]\right)}{\partial\alpha_i\partial\phi}\right\} = \mathrm{E}\left\{V(n)\right\}\mathrm{E}\left\{A_i\left(U(n)\right)\right\}.$$

In addition, since $\Im\{x(n)\}$ is zero mean and $\boldsymbol{w}(n)$ is a zero-mean complex Gaussian random vector, then from (74) $V(n)$ is also zero mean. Hence, $[\boldsymbol{I}_{\text{NDA}}(\boldsymbol{\alpha})]_{1,3} = [\boldsymbol{I}_{\text{NDA}}(\boldsymbol{\alpha})]_{2,3} = 0$. We also have:

$$\begin{aligned}\dot{U}(n) &= 2\Re\left\{e^{j\phi}\ \boldsymbol{y}(n)^H\dot{\boldsymbol{a}}\right\},\\ &= S\left(x(n)^*\boldsymbol{a}^H\dot{\boldsymbol{a}} + x(n)\dot{\boldsymbol{a}}^H\boldsymbol{a}\right) + \\ &\quad e^{j\phi}\boldsymbol{w}(n)^H\dot{\boldsymbol{a}} + e^{-j\phi}\dot{\boldsymbol{a}}^H\boldsymbol{w}(n).\end{aligned}$$

Using $\dot{\boldsymbol{a}}^H\boldsymbol{a} + \boldsymbol{a}^H\dot{\boldsymbol{a}} = 0$ derived from $||\boldsymbol{a}||^2 = M$, we obtain:

$$\dot{U}(n) = S\dot{\boldsymbol{a}}^H\boldsymbol{a}\Im\{x(n)\} + e^{j\phi}\boldsymbol{w}(n)^H\dot{\boldsymbol{a}} + e^{-j\phi}\dot{\boldsymbol{a}}^H\boldsymbol{w}(n). \quad (68)$$

Now, from (73) and (68), it can be shown that $U(n)$ and $\dot{U}(n)$ are independent with $\mathrm{E}\left\{\dot{U}(n)\right\} = 0$. Therefore, $[\boldsymbol{I}_{\text{NDA}}(\boldsymbol{\alpha})]_{1,4} = [\boldsymbol{I}_{\text{NDA}}(\boldsymbol{\alpha})]_{2,4} = 0$. Consequently, the NDA FIM is a block diagonal matrix.

Appendix B - Factorization of $D_{\boldsymbol{\alpha}}(n)$

We denote by $\tilde{\mathcal{C}} = \{\tilde{c}_1, \tilde{c}_2, \cdots, \tilde{c}_{2^{2(p-1)}}\}$ the subset of the alphabet points that lie in the top right quadrant of the constellation, i.e., $\tilde{\mathcal{C}} = \{(2i-1)d_p + j(2k-1)d_p\}_{i,k=1,2,\cdots,2^{p-1}}$. Then, we have $\mathcal{C} = \tilde{\mathcal{C}} \cup (-\tilde{\mathcal{C}}) \cup \tilde{\mathcal{C}}^* \cup (-\tilde{\mathcal{C}}^*)$. Consequently,

(12) is rewritten as follows:

$$D_{\boldsymbol{\alpha}}(n) = \sum_{\tilde{c}_k\in\tilde{\mathcal{C}}} \exp\left\{-\frac{S^2M|\tilde{c}_k|^2}{\sigma^2}\right\} \times \Bigg(\exp\left\{\frac{2S\Re\{\boldsymbol{y}(n)^H e^{j\phi}\tilde{c}_k\boldsymbol{a}\}}{\sigma^2}\right\} + \exp\left\{\frac{2S\Re\{\boldsymbol{y}(n)^H e^{j\phi}(-\tilde{c}_k)\boldsymbol{a}\}}{\sigma^2}\right\} + \exp\left\{\frac{2S\Re\{\boldsymbol{y}(n)^H e^{j\phi}\tilde{c}_k^*\boldsymbol{a}\}}{\sigma^2}\right\} + \exp\left\{\frac{2S\Re\{\boldsymbol{y}(n)^H e^{j\phi}(-\tilde{c}_k^*)\boldsymbol{a}\}}{\sigma^2}\right\} \Bigg). \quad (69)$$

Then using the fact that $e^x + e^{-x} = 2\cosh(x)$, we obtain:

$$D_{\boldsymbol{\alpha}}(n) = 2\sum_{\tilde{c_k}\in\tilde{\mathcal{C}}} \exp\left\{-\frac{S^2M|\tilde{c_k}|^2}{\sigma^2}\right\} \times \left(\cosh\left(\frac{2S\Re\{\boldsymbol{y}(n)^H e^{j\phi}\tilde{c}_k\boldsymbol{a}\}}{\sigma^2}\right) + \cosh\left(\frac{2S\Re\{\boldsymbol{y}(n)^H e^{j\phi}\tilde{c_k}^*\boldsymbol{a}\}}{\sigma^2}\right)\right). \quad (70)$$

Moreover, we have $\cosh(x) + \cosh(y) = 2\cosh(\frac{x+y}{2})\cosh(\frac{x-y}{2})$, and using the fact that $\tilde{c}_k + \tilde{c}_k^* = 2\Re\{\tilde{c}_k\}$ and $\tilde{c}_k - \tilde{c}_k^* = 2j\Im\{\tilde{c}_k\}$, (70) is rewritten as follows:

$$\begin{aligned} D_{\boldsymbol{\alpha}}(n) &= 4\sum_{\tilde{c_k}\in\tilde{\mathcal{C}}} \exp\left\{-\frac{S^2M|\tilde{c_k}|^2}{\sigma^2}\right\} \times \cosh\left(\frac{2S\Re\{\tilde{c}_k\}\Re\{\boldsymbol{y}(n)^H e^{j\phi}\boldsymbol{a}\}}{\sigma^2}\right) \times \cosh\left(\frac{2S\Im\{\tilde{c}_k\}\Im\{\boldsymbol{y}(n)^H e^{j\phi}\boldsymbol{a}\}}{\sigma^2}\right), \quad (71)\\ &= 4\sum_{i=1}^{2^{p-1}}\sum_{k=1}^{2^{p-1}} \exp\left\{-\frac{S^2M((2i-1)^2+(2k-1)^2)d_p^2}{\sigma^2}\right\} \times \cosh\left(\frac{2S(2i-1)d_p\Re\{\boldsymbol{y}(n)^H e^{j\phi}\boldsymbol{a}\}}{\sigma^2}\right) \times \cosh\left(\frac{2S(2k-1)d_p\Im\{\boldsymbol{y}(n)^H e^{j\phi}\boldsymbol{a}\}}{\sigma^2}\right). \quad (72)\end{aligned}$$

Finally, after splitting the two sums in (72), we obtain the expression of $D_{\boldsymbol{\alpha}}(n)$ given by (15).

APPENDIX C - PROOF OF THE INDEPENDENCE OF U(N) AND V(N)

We have:

$$\begin{aligned} U(n) &= 2\,\Re\{e^{j\phi}\ \boldsymbol{y}(n)^H\boldsymbol{a}\},\\ V(n) &= 2\,\Im\{e^{j\phi}\ \boldsymbol{y}(n)^H\boldsymbol{a}\}.\end{aligned}$$

Since $\boldsymbol{y}(n) = S\ e^{j\phi}\ x(n)\ \boldsymbol{a} + \boldsymbol{w}(n)$, we show that:

$$U(n) = 2SM\Re\{x(n)\} + e^{j\phi}\boldsymbol{w}(n)^H\boldsymbol{a} + e^{-j\phi}\boldsymbol{a}^H\boldsymbol{w}(n), \quad (73)$$

$$V(n) = j\left[2jSM\Im\{x(n)\} - e^{j\phi}\boldsymbol{w}(n)^H\boldsymbol{a} + e^{-j\phi}\boldsymbol{a}^H\boldsymbol{w}(n)\right]. \quad (74)$$

As previously assumed in [8], we take into account the hypothesis of the independence of $\Re\{x(n)\}$ and $\Im\{x(n)\}$. We also verify that $x(n)$ and the couple $(e^{j\phi}\boldsymbol{w}(n)^H\boldsymbol{a} + e^{-j\phi}\boldsymbol{a}^H\boldsymbol{w}(n), \frac{1}{j}(e^{j\phi}\boldsymbol{w}(n)^H\boldsymbol{a} - e^{-j\phi}\boldsymbol{a}^H\boldsymbol{w}(n)))$ are independent and that $x(n)^*$ and $(e^{j\phi}\boldsymbol{w}(n)^H\boldsymbol{a} + e^{-j\phi}\boldsymbol{a}^H\boldsymbol{w}(n), \frac{1}{j}(e^{j\phi}\boldsymbol{w}(n)^H\boldsymbol{a} - e^{-j\phi}\boldsymbol{a}^H\boldsymbol{w}(n)))$ are independent as well. Moreover, the two terms of the couple $(e^{j\phi}\boldsymbol{w}(n)^H\boldsymbol{a} + e^{-j\phi}\boldsymbol{a}^H\boldsymbol{w}(n)$ and $\frac{1}{j}(e^{j\phi}\boldsymbol{w}(n)^H\boldsymbol{a} - e^{-j\phi}\boldsymbol{a}^H\boldsymbol{w}(n)))$ are uncorrelated Gaussian random variables and therefore independent. Hence, $U(n)$ and $V(n)$ are independent.

APPENDIX D - PROOF OF (22) AND (23)

We have from (1)

$$\boldsymbol{y}(n) = S\ e^{j\phi}\ \boldsymbol{a}\ x(n) + \boldsymbol{w}(n), \quad n = 1, 2, \ldots, N.$$

Then, we obtain

$$2e^{j\phi}\ \boldsymbol{y}(n)^H\boldsymbol{a} = 2SMx(n)^* + 2e^{j\phi}\boldsymbol{w}(n)^H\boldsymbol{a}.$$

Otherwise, $2e^{j\phi}\boldsymbol{w}(n)^H\boldsymbol{a}$ can be written as

$$2e^{j\phi}\boldsymbol{w}(n)^H\boldsymbol{a} = 2e^{j\phi}\sum_{i=1}^{M} w_i(n)^* a_i,$$

where $w_i(n)$ and a_i are the i'th components of $\boldsymbol{w}(n)$ and $\boldsymbol{a}$, respectively. Therefore, $2e^{j\phi}\boldsymbol{w}(n)^H\boldsymbol{a}$ is a zero-mean Gaussian random variable whose variance is given by

$$\begin{aligned} \mathrm{var}(2e^{j\phi}\boldsymbol{w}(n)^H\boldsymbol{a}) &= 4\mathrm{var}\left(e^{j\phi}\sum_{i=1}^{M} w_i(n)^* a_i\right),\\ &= 4\,\mathrm{E}\left\{\left(\sum_{i=1}^{M} w_i(n)^* a_i\right)\left(\sum_{k=1}^{M} w_k(n) a_i^*\right)\right\},\\ &= 4\,\mathrm{E}\left\{\sum_{i=1}^{M} |a_i|^2 w_i(n)^* w_i(n)\right\},\\ &= 4\sum_{i=1}^{M}\mathrm{E}\{|a_i|^2\}\,\mathrm{E}\{w_i(n)^* w_i(n)\}\\ &= 4M\sigma^2.\end{aligned}$$

Therefore, the pdf of the random variable $2e^{j\phi}\boldsymbol{w}(n)^H\boldsymbol{a}$ is written as follows

$$P[2e^{j\phi}\boldsymbol{w}(n)^H\boldsymbol{a}] = \frac{1}{4\pi M\sigma^2}\exp\left\{-\frac{|2e^{j\phi}\boldsymbol{w}(n)^H\boldsymbol{a}|^2}{4M\sigma^2}\right\}. \quad (75)$$

Consequently, we obtain the pdf of the random variable $\vartheta(n) = 2\ e^{j\phi}\ \boldsymbol{y}(n)^H\boldsymbol{a}$

$$\begin{aligned} P[\vartheta(n)|x(n)] &= \frac{\exp\left\{-\frac{|2e^{j\phi}\boldsymbol{y}(n)^H\boldsymbol{a}-2SMx(n)|^2}{4M\sigma^2}\right\}}{4\pi M\sigma^2},\\ &= \frac{1}{4\pi M\sigma^2}\exp\left\{-\frac{U(n)^2+V(n)^2}{4M\sigma^2}\right\} \times \exp\left\{-\frac{S^2M|x(n)|^2}{\sigma^2}\right\} \times \exp\left\{\frac{2S\Re\{e^{j\phi}x(n)\boldsymbol{y}(n)^H\boldsymbol{a}\}}{\sigma^2}\right\}, \quad (76)\end{aligned}$$

where $U(n)$ and $V(n)$ are given by (17) and (18), respectively. Averaging (76) with respect to the symbols $x(n)$, we obtain the following result

$$P[\vartheta(n);\boldsymbol{\alpha}] = \frac{1}{4\pi LM\sigma^2}\exp\left\{-\frac{U(n)^2+V(n)^2}{4M\sigma^2}\right\}D_{\boldsymbol{\alpha}}(n), \tag{77}$$

where $D_{\boldsymbol{\alpha}}(n)$ is given by (12). We also have from (15) that $D_{\boldsymbol{\alpha}}(n)$ is factorized as follows:

$$D_{\boldsymbol{\alpha}}(n) = 4F_{\boldsymbol{\alpha}}(U(n))F_{\boldsymbol{\alpha}}(V(n)), \tag{78}$$

where

$$F_{\boldsymbol{\alpha}}(t)=\sum_{i=1}^{2^{p-1}}\exp\left\{-\frac{S^2M(2i-1)^2d_p^2}{\sigma^2}\right\}\cosh\left(\frac{(2i-1)d_pSt}{\sigma^2}\right).$$

Therefore, from (15), (21) and (77), we obtain the expressions of $P[U(n);\boldsymbol{\alpha}]$ and $P[V(n);\boldsymbol{\alpha}]$ given by (22) and (23), respectively.

Appendix E - Derivation of $\ddot{U}(n)$ and Proof of the Independence of $U(n)$ and $z(n)$

We have:

$$\ddot{U}(n)=\frac{\partial\dot{U}(n)}{\partial\theta}=2\Re\left\{e^{j\phi}\ \boldsymbol{y}(n)^H\ddot{\boldsymbol{a}}\right\}.$$

Using $\boldsymbol{y}(n)=S\ e^{j\phi}\ x(n)\ \boldsymbol{a}+\boldsymbol{w}(n)$, we show that:

$$\ddot{U}(n)=S\left(x(n)^*\boldsymbol{a}^H\ddot{\boldsymbol{a}}+x(n)\ddot{\boldsymbol{a}}^H\boldsymbol{a}\right)+ e^{j\phi}\boldsymbol{w}(n)^H\ddot{\boldsymbol{a}}+e^{-j\phi}\ddot{\boldsymbol{a}}^H\boldsymbol{w}(n). \tag{79}$$

Then, replacing $x(n)$ by $\frac{U(n)-jV(n)}{2MS}-\frac{e^{-j\phi}\ \boldsymbol{w}(n)^H\boldsymbol{a}}{MS}$ in (79) and using the identity $\ddot{\boldsymbol{a}}^H\boldsymbol{a}+\boldsymbol{a}^H\ddot{\boldsymbol{a}}+2||\dot{\boldsymbol{a}}||^2=0$ derived from $||\boldsymbol{a}||^2=M$, we obtain:

$$\ddot{U}(n) = -\frac{||\dot{\boldsymbol{a}}||^2}{M}U(n)+\frac{j}{2M}\left(\boldsymbol{a}^H\ddot{\boldsymbol{a}}-\ddot{\boldsymbol{a}}^H\boldsymbol{a}\right)V(n)+z(n),$$

with

$$z(n)=e^{j\phi}\boldsymbol{w}(n)^H\ddot{\boldsymbol{a}}+e^{-j\phi}\ddot{\boldsymbol{a}}^H\boldsymbol{w}(n)- \frac{1}{M}\left(e^{j\phi}\boldsymbol{w}(n)^H\boldsymbol{a}\boldsymbol{a}^H\ddot{\boldsymbol{a}}+e^{-j\phi}\boldsymbol{a}^H\boldsymbol{w}(n)\ddot{\boldsymbol{a}}^H\boldsymbol{a}\right).$$

By denoting $z'(n) = e^{j\phi}\ \boldsymbol{w}(n)^H\boldsymbol{a}+e^{-j\phi}\ \boldsymbol{a}^H\boldsymbol{w}(n)$ and $z''(n)=e^{j\phi}\ \boldsymbol{w}(n)^H\ddot{\boldsymbol{a}}+e^{-j\phi}\ \ddot{\boldsymbol{a}}^H\boldsymbol{w}(n)$, we verify that $z(n)$, $z'(n)$ and $z''(n)$ are zero-mean Gaussian distributed with

$$\begin{aligned}\mathrm{E}\left\{z'(n)z(n)\right\} &= \mathrm{E}\left\{z'(n)z''(n)\right\}- \frac{1}{M}\mathrm{E}\Big\{z'(n)(e^{j\phi}\boldsymbol{w}(n)^H\boldsymbol{a}\boldsymbol{a}^H\ddot{\boldsymbol{a}}+ e^{-j\phi}\ \boldsymbol{a}^H\boldsymbol{w}(n)\ddot{\boldsymbol{a}}^H\boldsymbol{a})\Big\},\\ &= -2||\dot{\boldsymbol{a}}||^2\ \sigma^2-\frac{1}{M}(-2M||\dot{\boldsymbol{a}}||^2\ \sigma^2)\\ &= 0.\end{aligned}$$

Consequently, $z'(n)$ and $z(n)$ are independent random variables. Since $\Re\{x(n)\}$ and $z(n)$ are also independent, $U(n)$ and $z(n)$ are independent.

Appendix F - Proof of (33)

We have:

$$\mathrm{E}\left\{\ddot{U}(n)\delta_{\boldsymbol{\alpha},1}\left(U(n)\right)\right\} = -\frac{||\dot{\boldsymbol{a}}||^2}{M}\int_{-\infty}^{+\infty}U(n)\delta_{\boldsymbol{\alpha},1}\left(U(n)\right)\times P[U(n);\boldsymbol{\alpha}]dU(n),$$

where

$$\int_{-\infty}^{+\infty}U(n)\delta_{\boldsymbol{\alpha},1}\left(U(n)\right)P[U(n);\boldsymbol{\alpha}]dU(n) = \frac{Sd_p}{\sqrt{\pi M}2^p\sigma^3}\times \sum_{i=1}^{2^{p-1}}e^{-\frac{MS^2d_p^2(2i-1)^2}{\sigma^2}}(2i-1)\Gamma_{\boldsymbol{\alpha}}^{(i)}, \tag{80}$$

with

$$\begin{aligned}\Gamma_{\boldsymbol{\alpha}}^{(i)} &= \int_{-\infty}^{+\infty}U(n)\ e^{-\frac{U(n)^2}{4M\sigma^2}}\sinh\left(\frac{(2i-1)d_pS}{\sigma^2}U(n)\right)dU(n),\\ &= 4MS\sigma\sqrt{\pi M}d_p(2i-1)\exp\left\{\frac{MS^2d_p^2(2i-1)^2}{\sigma^2}\right\}.\end{aligned} \tag{81}$$

Hence, injecting (81) in (80) we obtain:

$$\mathrm{E}\left\{\ddot{U}(n)\delta_{\boldsymbol{\alpha},2}\left(U(n)\right)\right\} = -\frac{4||\dot{\boldsymbol{a}}||^2S^2d_p^2}{2^p\sigma^2}A_{2,p}.$$

Appendix G - Other Algebraic Manipulations to Derive the NDA FIM

Using the regularity condition

$$\frac{\partial}{\partial\alpha_k}\int P[\boldsymbol{y}(n);\boldsymbol{\alpha}]d\boldsymbol{y}(n)=\int\frac{\partial P[\boldsymbol{y}(n);\boldsymbol{\alpha}]}{\partial\alpha_k}d\boldsymbol{y}(n),$$

which is fulfilled for any finite mixtures of Gaussian distributions, we have $\mathrm{E}\left\{\frac{\partial\ln(P[\boldsymbol{y}(n);\boldsymbol{\alpha}])}{\partial\sigma}\right\}=0$. Furthermore, we have

$$\mathrm{E}\left\{\frac{\partial\ln\left(P[\boldsymbol{y}(n);\boldsymbol{\alpha}]\right)}{\partial\sigma}\right\} = -\frac{2M}{\sigma}+\frac{2}{\sigma^3}\mathrm{E}\left\{||\boldsymbol{y}(n)||^2\right\}+ 2\,\mathrm{E}\left\{\frac{\partial\ln\left(F_{\boldsymbol{\alpha}}(U(n))\right)}{\partial\sigma}\right\}.$$

Thus, we obtain:

$$\mathrm{E}\left\{||\boldsymbol{y}(n)||^2\right\}=M\ \sigma^2-\sigma^3\ \mathrm{E}\left\{\frac{\partial\ln\left(F_{\boldsymbol{\alpha}}(U(n))\right)}{\partial\sigma}\right\}.$$

This identity enables us to derive the term $[\boldsymbol{I}_1^{\mathrm{NDA}}(\sigma,S)]_{1,1}$.

To derive $[\boldsymbol{I}_2^{\mathrm{NDA}}(\phi,\theta)]_{1,2}$, we show that $\dot{U}(n),V(n)$ and $U(n)$ are independent and hence we have:

$$\mathrm{E}\left\{\dot{U}(n)V(n)\right\}=-j\dot{\boldsymbol{a}}^H\boldsymbol{a}\left(2S^2\ M+2\ \sigma^2\right).$$

Moreover, we consider

$$\dot{V}(n)=\frac{\partial V(n)}{\partial\theta} = 2\Im\{e^{j\phi}\boldsymbol{y}(n)^H\dot{\boldsymbol{a}}\}=\frac{j\dot{\boldsymbol{a}}^H\boldsymbol{a}}{M}\ U(n)+r(n),$$

with

$$r(n) = \frac{j}{M}(e^{j\phi}\ \boldsymbol{w}(n)^H\boldsymbol{a}\boldsymbol{a}^H\dot{\boldsymbol{a}}-e^{-j\phi}\ \dot{\boldsymbol{a}}^H\boldsymbol{w}(n)\dot{\boldsymbol{a}}^H\boldsymbol{a})- j(e^{j\phi}\ \boldsymbol{w}(n)^H\dot{\boldsymbol{a}}-e^{-j\phi}\ \dot{\boldsymbol{a}}^H\boldsymbol{w}(n)).$$

In addition, it can be seen that $r(n)$ is zero-mean Gaussian distributed with $\mathrm{E}\{z(n)r(n)\}=0$.

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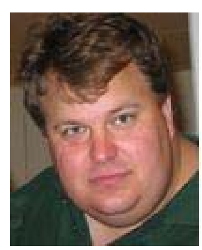

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