Second-Order Moment-Based Direction Finding of a Single Source for ULA systems

Faouzi Bellili, Sofiène Affes, and Alex Stéphenne

INRS-EMT, 800, de la Gauchetière Ouest, Bureau 6900, Montreal, Qc, H5A 1K6, Canada
Emails: bellili@emt.inrs.ca, affes@emt.inrs.ca, and stephenne@ieee.org

***Abstract*— We address the problem of direction of arrival (DOA) finding for uniform linear arrays (ULAs). We derive a new and very simple method for estimating the DOA of a single source based on the covariance matrix of the received signal. The new method is non-data-aided (NDA) and does not therefore impinge on the whole throughput of the system. The noise components are assumed spatially and temporally white. The new method is derived in closed form and it exhibits exactly — over a wide practical SNR range — the same performance of the popular root-MUSIC algorithm, a powerful DOA estimation technique for ULA configurations. Therefore, the new estimator offers a way for a rapid and very easy DOA evaluation; making it very attractive for practical implementation as compared to the root-MUSIC algorithm that relies on the heavy operation of eigen decomposition.**

I. Introduction

From radar to sonar and cellular wireless communications, direction of arrival estimation has attracted a lot of interest during the last decades; and intensive research works have been conducted on this hot topic. Many DOA estimators were derived over the years and, roughly speaking, they can be categorized into two major categories: inphase/quadrature (I/Q)-based algorithms and moment-based algorithms. The first category includes in particular the maximum likelihood (ML)-based approaches such as the ML techniques introduced in [1], the decoupled maximum likelihood (DEML) angle estimator [2] and the modified likelihood (MML) function approach [3]. These estimators that solve the ML criterion provide usually very accurate DOA estimates but they suffer from a very high computational burden; making them even impossible to implement in practical situations where the computational cost is a crucial design parameter. This is for example the case of mobile phones where the complexity should be kept at the lowest possible level in order to increase the battery autonomy. To circumvent this challenging problem, the ML estimators can be derived in closed form in the data-aided DA case where a training sequence (a sequence that is perfectly known to the receiver) is periodically transmitted. But, in counterpart, this training overhead has the major drawback of limiting the whole throughput of the system. Therefore, there has been a need for deriving completely blind (or NDA) estimators which should also be easily implemented. In this context, an interesting class of the so-called moment-based estimators which base the estimation process only on the envelope of the received signal (blind) were derived in the literature. They are much less computationally demanding than the NDA ML-based approaches. These include the multiple signal classification (MUSIC) estimator [4], estimation of signal parameters via rotational invariance technique (ESPRIT) [5] and the root-MUSIC algorithm [6]. These pioneering high-resolution techniques, which were initially formulated in the context of direction finding, have been also successfully applied in many other applications in the field of signal processing.

In particular, the root-MUSIC algorithm is a powerful estimation technique especially for ULA systems and it provides very accurate estimates even for low SNR values. Surely, it is simpler than the ML-based technique, but still it requires a complex operation of eigen decomposition whose complexity increases substantially with the array size. Thus, avoiding this computationally demanding operation is essential for the purpose of power saving in situations of scarce energy; such as satellite communications where the solar energy is the only source of power or wireless sensor networks where nodes even rely sometimes on energy harvesting.

Motivated by these facts, in this paper we derive a new moment-based technique for DOA estimation of a single source signal impinging on a receiver equipped with a ULA of receiving antenna elements. The single source case is encountered, for instance, when dealing with CDMA signals after performing the despreading operation. Indeed, one single source corresponding to the desired signal will be preserved through constructive correlation and any other source will be dramatically reduced by destructive correlation and incorporated in the noise component. The single source model can also be obtained in case of multi-source transmissions for which an algorithm of blind source separation (BSS) is applied as a post treatment. Afterward, the observation obtained for each source will follow a single-source model; for which our newly derived method can be adequately applied to estimate the DOA of corresponding source.

The new method exhibits exactly the same performance — over a wide range of practical SNRs — of the root-MUSIC estimator. Yet, it is derived in closed form and hence it is much easier to implement in practice and also has a much lower computational cost (no eigen decomposition). In other words, the new method can be thought of as the closed-form version of the root-MUSIC estimator when a single source is active.

The rest of this paper will be organized as follows. In section II, we will introduce the system model that will be used throughout the article. In section III, we will derive the new

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DOA estimator. In section IV, we will assess the performance of the new estimation technique through Monte-Carlo simulations and concluding remarks will be drawn out in section V.

II. System Model

We consider a uniform linear array (ULA) with N_a antenna elements receiving a planar wave impinging from a single source. The transmitted symbols are assumed to be independent and equally likely drawn from any M-ary constellation. Further, assuming perfect frequency synchronization, the received signal on the $\{i^{th}\}_{i=1}^{N_a}$ antenna element, at the output of the matched filter, can be modeled as a complex signal as follows[1]:

$$y_i(n) = he^{j(i-1)\pi\sin(\theta)}a(n) + w_i(n), \quad i = 1, 2, \cdots, N_a \quad (1)$$

where j is the complex number verifying $j^2 = -1$. Moreover, at time index n, $a(n)$ is the transmitted symbols, and $w_i(n)$ is the noise component on the i^{th} antenna branch that is modelled by a zero-mean complex Gaussian random variable with independent real and imaginary parts, each of variance σ^2. h is the complex channel coefficient (i.e., $h = |h|e^{j\phi}$ with ϕ standing for any possible channel distortion phase). Moreover θ is the unknown DOA of the wave impinging from the far-field source. We assume hereafter that the noise components $\boldsymbol{w}(n) = [w_1(n),\ w_2(n), \cdots,\ w_{N_a}(n)]^T$ are spatially uncorrelated, i.e., $E\{\boldsymbol{w}(n)^H\boldsymbol{w}(n)\} = \sigma^2\boldsymbol{I}_{N_a}$ where $\boldsymbol{I}_{N_a}$ is the $N_a \times N_a$ identity matrix. We also assume that the energy of the transmitted signal is normalized to one, i.e., $E\{|a(n)|^2\} = 1$). Then we define the SNR of the system as follows:

$$\rho = \frac{\mathrm{E}\{|h|^2|a(n)|^2\}}{2\sigma^2} = \frac{|h|^2}{2\sigma^2}. \quad (2)$$

III. Formulation of the New Moment-Based DOA Estimator

The new estimator is primarily based on the second-order cross moments of the received signals on the antenna array. In fact, the cross-moment between the received signals on the i^{th} and the k^{th} receiving antenna elements, is given by:

$$M_\theta(i,k) = E\{y_i(n)y_k^*(n)\},\ i,k = 1, 2, \cdots, N_a. \quad (3)$$

Then using the fact that the transmitted symbols are iid and the noise components are spatially and temporally white, $M_{\boldsymbol{\theta}}(i,k)$ reduces simply to:

$$M_\theta(i,k) = \begin{cases} |h|^2 + 2\sigma^2, & \text{for } i = k \\ |h|^2 e^{j\pi(i-k)\sin(\theta)}, & \text{for } i \neq k \end{cases} \quad (4)$$

Actually, $\{M_{\boldsymbol{\theta}}(i,k)\}_{i,k=1}^{N_a}$ are the entries, $[\bar{\boldsymbol{M}}(\boldsymbol{\theta})]_{i,k}$, of the autocovariance matrix, $\bar{\boldsymbol{M}}(\boldsymbol{\theta})$, of the received signal $\boldsymbol{y}(n) = [y_1(n),\ y_2(n), \cdots,\ y_{N_a}(n)]^T$ which is defined as $\bar{\boldsymbol{M}}(\boldsymbol{\theta}) = E\{\boldsymbol{y}(n)\boldsymbol{y}^H(n)\}$.
Next, since $\bar{\boldsymbol{M}}(\boldsymbol{\theta})$ is a Hermetian matrix, we will only consider the elements contained in the lower triangular matrix obtained from $\bar{\boldsymbol{M}}(\boldsymbol{\theta})$. Therefore, from now on, the counters i and k always verify $i > k$. Actually, the unknown DOA will be estimated from the phases of the estimated cross-moments, $\{\widehat{M}_\theta(i,k)\}_{i>k}$, which are found by simple sample averaging as follows:

$$\widehat{M}_\theta(i,k) = \frac{1}{N}\sum_{n=1}^{N} y_i(n)y_k^*(n), \quad i = 1, 2, \cdots, N_a, \quad k \leq i. \quad (5)$$

First, we notice from (4) that the cross-moments $M_{\boldsymbol{\theta}}(i,k)$ that belong to the same diagonal have the same expression, i.e., whenever $i - k = l$ for $l = 1, 2, \cdots, N_a - 1$, we have:

$$\begin{aligned} M_l(\theta) &= M_\theta(i+l, i) \quad i = 1, 2, \cdots, N_a - l, \\ &= |h|^2 e^{j\pi l\sin(\theta)}. \end{aligned} \quad (6)$$

Therefore, $\widehat{M}_\theta(i,k)$ can be averaged over all the pairs (i,k) verifying $i - k = l$ to obtain a more refined set of moments, $\{\widehat{M}_l(\theta)\}_{l=1}^{N_a-1}$, as follows:

$$\widehat{M}_l(\theta) = \frac{1}{N_a - l}\sum_{i-k=l} \widehat{M}_\theta(i,k), \quad l = 1, 2, \cdots, N_a - 1. \quad (7)$$

Then the more accurate cross-moments $\widehat{M}_l(\theta)$ (compared to the elementary cross-moments $M_{\boldsymbol{\theta}}(i,k)$) can be used to obtain a set of more accurate estimates, $\{\widehat{\theta}\}_{l=1}^{N_a-1}$, of the unknown DOA. In fact, we define the following statistic:

$$\alpha_l(\theta) = \arg\left(M_l(\theta)\right), \quad (8)$$

where $\arg(.)$ returns the argument or the angle of any complex number, respectively. Note here that an accurate estimate, $\widehat{\alpha}_l(\theta)$, of $\alpha_l(\theta)$ can be obtained as:

$$\widehat{\alpha}_l(\theta) = \arg\left(\widehat{M}_l(\theta)\right), \quad (9)$$

from which an estimate, $\widehat{\theta}_l$, of θ can be obtained for each $l = 1, 2, \cdots, N_a$. However, it should be kept in mind that the function $\arg(.)$ as used in (9) will not always return the quantity $\pi l\sin(\theta)$ that appears in the argument of $\widehat{M}_l(\theta)$. This is because $\pi l\sin(\theta)$ may not always be confined in $[-\pi, \pi]$. Indeed, this is only true for $l = 1$ since $-1 < \sin(\theta) < 1$ and therefore $-\pi < \pi\sin(\theta) < \pi$. Yet, this initial estimate obtained in the special case of $l = 1$ can be used to obtain other estimates for $l = 2, 3, \cdots, N_a - 1$, as explained subsequently.
To begin with, it is easy to see that $\alpha_1(\theta) = \pi\sin(\theta)$. Thus, the first estimate is simply obtained from $\widehat{\alpha}_1(\theta)$ as follows:

$$\widehat{\theta}_1 = \arcsin\left(\frac{\widehat{\alpha}_1(\theta)}{\pi}\right). \quad (10)$$

Now having $\widehat{\theta}_1$ at hand, we check for each $l = 2, 3, \cdots, N_a - 1$, if $|l\sin(\widehat{\theta}_1)| < 1$. If yes, then for this l we have $\alpha_l(\theta) = \pi l\sin(\theta)$ and the corresponding estimate $\widehat{\theta}_l$ is simply obtained as follows:

$$\widehat{\theta}_l = \arcsin\left(\frac{\widehat{\alpha}_l(\theta)}{l\pi}\right). \quad (11)$$

If not, i.e., $|l\sin(\theta_1)| = |l\sin(\widehat{\theta}_1)| > 1$ (by assuming the true DOA value is $\theta = \widehat{\theta}_1$), then $\alpha_l(\theta) \neq \pi l\sin(\theta)$ and

[1] Note here that receiving antenna elements are supposed to be spaced by half the wavelength, i.e., $d = \lambda/2$ where d is the distance between two consecutive antenna branches and λ is the wavelength.

the estimation procedure should then be carefully handled as detailed subsequently.

In fact, in this situation, we have two cases depending on the sign of $l\sin(\theta)$, i.e., $l\sin(\theta) > 1$ or $l\sin(\theta) < -1$. In both cases, we decompose this quatity in a sum of an integer and a fraction, then the estimation process reduces to a task of calculating this fraction using the estimated moments.

A. Case 1: if $l\sin(\theta_1) = l\sin(\widehat{\theta}_1) > 1$:

In this case, we decompose $l\sin(\theta)$ as follows:

$$l\sin(\theta) = m + p, \tag{12}$$

where m is the largest positive integer verifying $m < l\sin(\widehat{\theta}_1)$ and p is a real remainder in $]0,1[$. Then:

1) If m is even, i.e., $m = 2r$ where $r \in \mathbb{N}$: In this case we have $\pi l\sin(\theta) = 2r\pi + p\pi$ which means that:

$$M_l(\theta) = |h|^2 e^{jp\pi} \tag{13}$$

and therefore $\alpha_l(\theta) = p\pi$ meaning that:

$$p = \frac{\alpha_l(\theta)}{\pi}, \tag{14}$$

Hence, injecting (14) in (12) it follows that $\pi l\sin(\theta) = 2r\pi + \alpha_l(\theta)$ from which the l^{th} estimate of θ is obtained as:

$$\widehat{\theta}_l = \arcsin\left(\frac{\widehat{\alpha}_l(\theta)}{\pi l} + \frac{m}{l}\right). \tag{15}$$

2) If m is odd, i.e., $m = 2r+1$ where $r \in \mathbb{N}$: In this case we have:

$$\pi l\sin(\theta) = 2r\pi + (1+p)\pi. \tag{16}$$

Consequently, it can be seen that the statistic $\alpha(\theta)$ is expressed as

$$\begin{aligned}\alpha_l(\theta) &= (1+p)\pi - 2\pi \\ &= (p-1)\pi.\end{aligned} \tag{17}$$

Now, from (17), we obtain the value of the fraction p involved in (16) as follows:

$$p = 1 + \frac{\alpha_l(\theta)}{\pi}. \tag{18}$$

Then, injecting (18) in (16) and resorting to simple algebraic manipulations, we obtain the l^{th} estimate as:

$$\widehat{\theta}_l = \arcsin\left(\frac{\widehat{\alpha}_l(\theta)}{\pi l} + \frac{m+1}{l}\right). \tag{19}$$

B. Case 2: if $l\sin(\theta) = l\sin(\widehat{\theta}_1) < -1$:

In this case, we write $l\sin(\theta)$ as follows:

$$l\sin(\theta) = m + q \tag{20}$$

where m is the closest (to $l\sin(\widehat{\theta}_1)$) negative integer that verifies $l\sin(\theta) = l\sin(\widehat{\theta}_1) < m \leq -1$ and q is a real negative fraction in $]-1,0[$. Then:

1) If m is even, i.e., $m = 2r$ where $r \in \mathbb{Z}^*_-$: in this case we have:

$$\pi l\sin(\theta) = 2r\pi + q\pi \tag{21}$$

and therefore $\widehat{\alpha}_l(\theta) = q\pi$. Then injecting the value of $q = \frac{\widehat{\alpha}_l(\theta)}{\pi}$ in (21), we obtain:

$$\widehat{\theta}_l = \arcsin\left(\frac{\widehat{\alpha}_l(\theta)}{\pi l} + \frac{m}{l}\right), \tag{22}$$

which is the same estimate obtained in (15)

2) If m is odd, i.e., $m = 2r-1$ where $r \in \mathbb{Z}_-$: in this case, we have:

$$\begin{aligned}\pi l\sin(\theta) &= (2r-1)\pi + q\pi \\ &= 2(r-1)\pi + (1+q)\pi\end{aligned} \tag{23}$$

Then, it can be seen that $\alpha_l(\theta) = (q+1)\pi$ from which the fraction q is obtained as $q = \frac{\widehat{\alpha}_l(\theta)}{\pi} - 1$. Afterward, injecting q in (23), yields:

$$\sin(\theta) = \frac{m-1}{l} + \frac{\widehat{\alpha}_l(\theta)}{\pi l}. \tag{24}$$

Finally, the estimate $\widehat{\theta}_l$ is deduced as:

$$\widehat{\theta}_l = \arcsin\left(\frac{\widehat{\alpha}_l(\theta)}{\pi l} + \frac{m-1}{l}\right). \tag{25}$$

which is a bit different from the one obtained in (19).

Finally, a more refined estimate of the unknown DOA can be obtained by averaging over all the $N_a - 1$ estimates, $\{\widehat{\theta}_l\}_{l=1}^{N_a-1}$, as follows:

$$\widehat{\theta} = \frac{1}{N_a - 1}\sum_{l=1}^{N_a-1}\widehat{\theta}_l. \tag{26}$$

IV. Simulation Results

In this section we assess the performance of the new estimator using the mean square error (MSE) as a performance measure. The MSE is computed for the estimator $\widehat{\theta}$ of the DOA, θ, as follows:

$$\text{MSE}(\widehat{\theta}) = \frac{1}{M_c}\sum_{q=1}^{M_c}(\hat{\theta}^{(q)} - \theta)^2, \tag{27}$$

where M_c is the number of Monte-Carlo simulations which is set to $M_c = 1000$ for all simulations and $\hat{\theta}^{(q)}$ is the estimate of θ during the q^{th} Monte-Carlo run. We also consider the root-MUSIC (RM) estimator as a benchmark for the assessment of our newly developed method. In fact, the RM estimator remains so far among the best DOA estimators (in terms of accuracy) for ULA configurations.

In Fig. 1, we plot the estimation error for the two estimators for a true DOA value of $\theta = 0.3\pi$ in the presence of 4 receiving antenna elements. We see that the two estimators exhibit exactly the same performance over the entire practical SNR range of $[+2,\ 20]$ dB with a slight performance improvement for the root-MUSIC algorithm for the negative SNRs. Actually, this behavior holds regardles of the true DOA to be estimated. This can be more clearly seen from Fig. 2 where we plot the MSE

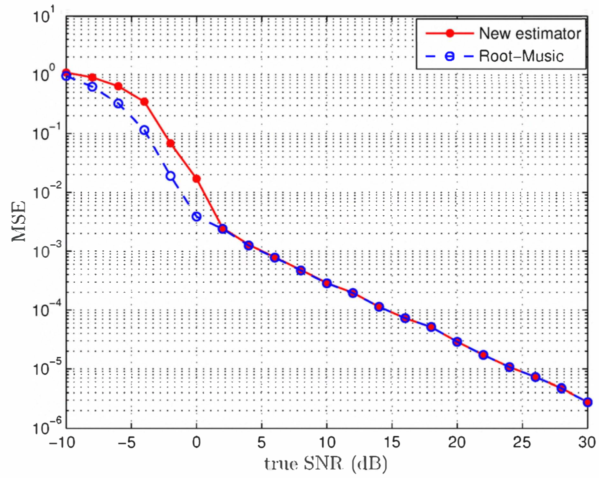

Fig. 1. MSE of the two estimators vs. the SNR, QPSK, $N_a = 4$, $N = 10$, $\theta = 0.2\pi$.

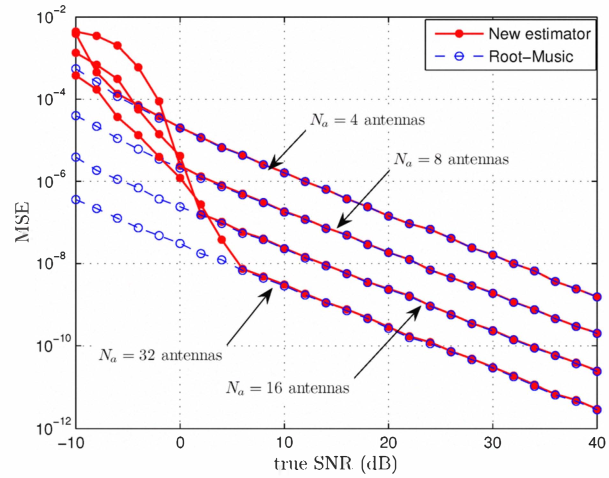

Fig. 3. MSE of the two estimators vs. the SNR for different array sizes, 16-QAM, $N = 1000$, $\theta = 0.3\pi$.

of the two estimators over the entire DOA range $]-\pi/2, \pi/2[$ for two SNR values, i.e., $SNR = 10$ dB and $SNR = 20$ dB. Furthermore, the performance improvement for the two

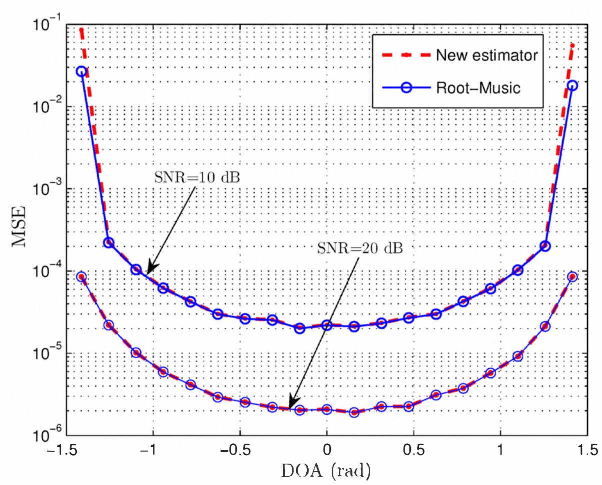

Fig. 2. MSE of the two estimators vs. the DOA for two different values of the DOA, 16-QAM, $N_a = 4$, $N = 50$.

estimators is also the same when the number of receiving antenna elements is increased. This is depicted in Fig. 3 where we plot the MSE for different values of $N_a = 4, 8, 16, 32$. In summary, we observe that, over a wide range of practical SNRs, the new simple estimator amounts to a closed-form version of the popular root-MUSIC estimator in the case of single-input multiple-output (SIMO) systems with a ULA receiving configuration. Yet, it involves very much simpler operations that can be more easily implemented in practice.

V. Conclusion

In this paper, we derived a new DOA estimation method for single planar wave impinging on a ULA antenna array. The noise components are assumed to be spatially and temporally white. The new method is NDA and based on the second-order cross-moments of the received signals. The new estimator is derived in closed-form and it exhibits the same statistical performance of the well-known root-MUSIC estimator. It is a very simple method and still provides very accurate estimates; making it well geared toward practical implementation for power-sensitive systems.

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