Subcarrier SNR ML Estimators and Cramér-Rao Bounds in Multicarrier Transmission

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***Abstract*—In this paper, considering a multicarrier transmissions system, we propose two techniques of maximum-likelihood (ML) subcarrier signal-to-noise ratio (SNR) estimation, in both data-aided (DA) and non-data-aided (NDA) schemes, and derive the corresponding Cramér-Rao lower bounds (CRLBs). The channel gains and phases are assumed to be constant over the observation window and the received signal is assumed to be corrupted by additive white Gaussian noise (AWGN). The proposed SNR estimators both exploit the mutual information between the different subcarriers and reach the corresponding CRLBs, as shown by Monte-Carlo simulations.**

I. Introduction

SNR estimation is a key parameter in communication systems, since a lot of applications in these systems require an accurate knowledge of the SNR, such as adaptive modulation [1], more so in wireless systems [2]. A common way to compare two unbiased SNR estimators is to compare their variances. An estimator outperforms another if its variance is lower. The CRLB is the best achievable variance for any unbiased estimator.

SNR estimators can be mainly categorized in two major categories, the non-data-aided (NDA) and data-aided (DA) estimators. Unlike DA SNR estimators, which base their estimation on an a priori knowledge of the transmitted data, NDA estimators base their estimation only on the received samples. SNR estimators can be also classified according to the way they process the data. Envelope-based SNR estimators exploit only the amplitude of the received signal. Those that use the whole information carried by the inphase and quadrature (I/Q) components of the signal are referred to as I/Q estimators.

As far as we know, the CRLBs have already been derived in closed-form or numerically computed [3], [4], [5] in single-carrier transmissions over AWGN channels, and for some particular constellations, like square QAM [6].

The estimators developed herein are the first that exploit the mutual information between the subcarriers in order to improve significantly the accuracy of the SNR estimates. The corresponding CRLBs have been previously derived in [7], [8], using the closed-form expressions for the Fisher information matrix (FIM) elements only available for square QAM constellations [6].

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The rest of the paper is organized as follows. The system model used in the rest of the paper is introduced in Section II. We derive the CRLBs for DA and NDA estimators in Section III. The two corresponding ML SNR estimators are derived in Section IV. Section V discloses our simulations results. We conclude in Section VI.

II. System Model

We consider a traditional digital communication system broadcasting and receiving any multicarrier signal. The channel gains and phases $\{S_k\}_{k=1,2,...,K}$ and $\{\phi_k\}_{k=1,2,...,K}$, respectively, are supposed to remain constant over the observation window for all K subcarriers. We assume also that we receive on the k^{th} subcarrier, an AWGN-corrupted signal with noise power $2\sigma_k^2$. Assuming perfect synchronization, the received signal at the input of the matched filter can be modeled as follows:

$$y_k(n) = S_k e^{j\phi_k} a_k(n) + w_k(n) \tag{1}$$

for $k = 1,2,...,K$ and $n = 1,2,...,N$, where, on the k^{th} tone and at time index n, $a_k(n)$ is the transmitted symbol and $y_k(n)$ is the corresponding received sample. $\{w_k(n)\}_{k=1,2,...,K,n=1,2,...,N}$ are the additive white noise components, which are modelled by complex zero-mean Gaussian random variables with independent real and imaginary parts, each of variance σ_k^2. N is the number of the received samples during the transmission interval and j is the imaginary number that verifies $j^2 = -1$. We denote by $\mathcal{C} = \{c_1, c_2, ..., c_M\}$ the alphabet of size M from which the transmitted symbols are drawn, and the constellation power is supposed to be normalized to one, i.e., $E\{|a_k(n)|^2\} = 1$, for $k = 1,2,...,K$. Moreover, we assume that a subset of the subcarriers experiences the same noise power $2\sigma^2$ (say, without loss of generality, the first L subcarriers, i.e., $\sigma_k^2 = \sigma^2$, $k = 1,2,...,L$).

We can now write the received signal in the following matrix form:

$$\mathbf{Y} = \left[\begin{array}{cccc} \mathbf{y}_1 & \mathbf{y}_2 & \cdots & \mathbf{y}_K \end{array}\right], \tag{2}$$

where

$$\mathbf{y}_k = \left[y_k(1), y_k(2), ..., y_k(N)\right]^T, \quad k = 1,2,...,K. \tag{3}$$

The transmitted data can be written in the same way:

$$\mathbf{A} = \left[\begin{array}{cccc} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_K \end{array}\right], \tag{4}$$

where

$$\mathbf{a}_k = [a_k(1), a_k(2), ..., a_k(N)]^T, \quad k = 1, 2, ..., K. \tag{5}$$

Based on the NK received samples, the subcarrier SNRs that we wish to estimate are defined as:

$$\rho_k = \begin{cases} \frac{S_k^2}{2\sigma^2}, & k = 1, 2, ..., L, \\ \frac{S_k^2}{2\sigma_k^2}, & k = L+1, L+2, ..., K. \end{cases} \tag{6}$$

The parameter vector to estimate is the following:

$$\begin{aligned} \mathbf{\Theta} &= [\theta_1, \theta_2, ..., \theta_{2K-L+1}]^T, & (7) \\ &= [\rho_1, \rho_2, ..., \rho_L, \sigma^2, \rho_{L+1}, \sigma_{L+1}^2, \rho_{L+2}, \\ &\quad \sigma_{L+2}^2, ..., \rho_K, \sigma_K^2]^T. & (8) \end{aligned}$$

Since the SNR is usually estimated over a decibel scale, we will use here the following parameter transformations:

$$g_k(\mathbf{\Theta}) = 10 \log_{10}(\rho_k), \quad k = 1, 2, ..., K. \tag{9}$$

We then define the parameter transformation vector:

$$\mathbf{g}(\mathbf{\Theta}) = [g_1(\mathbf{\Theta}), g_2(\mathbf{\Theta}), ..., g_K(\mathbf{\Theta})]. \tag{10}$$

III. SUBCARRIERS SNR CRLBS

In this section, under the assumptions made in the previous section, we will derive the subcarrier SNR CRLBs for any modulated signal. These CRLBs have been previously derived in [7], using the following parameter vector :

$$\widetilde{\mathbf{\Theta}} = \left[S_1, S_2, ..., S_L, \sigma^2, S_{L+1}, \sigma_{L+1}^2, S_{L+2}, \sigma_{L+2}^2, ..., S_K, \sigma_K^2\right]^T. \tag{11}$$

The CRLBs were derived there only for square QAM constellations, using the closed-form expressions derived in [6]. We derive here the CRLB numerically for any constellation.

Furthermore, we assume in the following that the channel phases have been perfectly recovered before SNR estimation, and we use the modified received samples, for $n = 1, 2, ..., N$ and $k = 1, 2, ..., K$:

$$\begin{aligned} \widetilde{y}_k(n) &= y_k(n) e^{-j\phi_k}, & (12) \\ &= S_k a_k(n) + w_k(n) e^{-j\phi_k}, & (13) \\ &= S_k a_k(n) + \widetilde{w}_k(n). & (14) \end{aligned}$$

Since $\widetilde{w}_k$ and w_k have the same statistics, we will use in the following w_k instead of $\widetilde{w}_k$ and y_k instead of $\widetilde{y}_k$, which amounts to assuming a zero phase offset.

Using an unbiased SNR estimator, and denoting the global log-likelihood function (LLF) by :

$$\Lambda(\mathbf{Y}|\mathbf{\Theta}) = \sum_{l=1}^{L} \Lambda(\mathbf{y}_l|\rho_l, \sigma^2) + \sum_{k=L+1}^{K} \Lambda(\mathbf{y}_k|\rho_k, \sigma_k^2), \tag{15}$$

the SNR CRLB satisfies:

$$\mathbf{CRLB}(\rho) = \frac{\partial \mathbf{g}(\mathbf{\Theta})}{\partial \mathbf{\Theta}} \mathbf{I}^{-1}(\mathbf{\Theta}) \frac{\partial \mathbf{g}(\mathbf{\Theta})^T}{\partial \mathbf{\Theta}}, \tag{16}$$

where $\mathbf{I}(\mathbf{\Theta})$ is the Fisher information matrix (FIM), defined as:

$$[\mathbf{I}(\mathbf{\Theta})]_{ij} = -E_{\mathbf{Y}} \left[\frac{\partial^2(\mathbf{Y}|\mathbf{\Theta})}{\partial \theta_i \partial \theta_j}\right], \tag{17}$$

and the matrix $\frac{\partial \mathbf{g}(\mathbf{\Theta})}{\partial \mathbf{\Theta}}$ is given by:

$$\left[\frac{\partial \mathbf{g}(\mathbf{\Theta})}{\partial \mathbf{\Theta}}\right]_{ij} = \frac{\partial g_i(\mathbf{\Theta})}{\partial \theta_j}. \tag{18}$$

Hence we obtain the same matrix forms found in [7], [8]:

$$\frac{\partial \mathbf{g}(\mathbf{\Theta})}{\partial \mathbf{\Theta}} = \begin{pmatrix} \mathbf{G}_1 & \mathbf{0}_{L\times 2(K-L)} \\ \mathbf{0}_{(K-L)\times(L+1)} & \mathbf{G}_2 \end{pmatrix}, \tag{19}$$

with

$$\mathbf{G}_1 = \begin{pmatrix} \frac{10}{\ln(10)\rho_1} & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & \frac{10}{\ln(10)\rho_L} & 0 \end{pmatrix}, \tag{20}$$

$$\mathbf{G}_2 = \begin{pmatrix} \frac{\partial g_{L+1}(\mathbf{\Theta}^{L+1})}{\partial \mathbf{\Theta}^{L+1}} & \mathbf{0}_{1\times 2} & \cdots & \mathbf{0}_{1\times 2} \\ \mathbf{0}_{1\times 2} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0}_{1\times 2} \\ \mathbf{0}_{1\times 2} & \cdots & \mathbf{0}_{1\times 2} & \frac{\partial g_K(\mathbf{\Theta}^K)}{\partial \mathbf{\Theta}^K} \end{pmatrix}, \tag{21}$$

where the elements of $\mathbf{\Theta}^k$ are defined by:

$$\mathbf{\Theta}^k = \begin{cases} \left[\rho_k, \sigma^2\right], k = 1, 2, ..., L, \\ \\ \left[\rho_k, \sigma_k^2\right], k = L+1, L+2, ..., K. \end{cases} \tag{22}$$

We will now use the following notations to derive the FIM:

$$b_k = -E_{\mathbf{y}_k}\left[\frac{\partial^2 \Lambda(\mathbf{y}_k|\mathbf{\Theta}^k)}{\partial \rho_k^2}\right], k = 1, 2, ..., K, \tag{23}$$

$$c_k = \begin{cases} -E_{\mathbf{y}_k}\left[\frac{\partial^2 \Lambda(\mathbf{y}_k|\mathbf{\Theta}^k)}{\partial \sigma^2 \partial \rho_k}\right], k = 1, 2, ..., L, \\ \\ -E_{\mathbf{y}_k}\left[\frac{\partial^2 \Lambda(\mathbf{y}_k|\mathbf{\Theta}^k)}{\partial \sigma_k^2 \partial \rho_k}\right], k = L+1, L+2, ..., K, \end{cases} \tag{24}$$

$$d_k = \begin{cases} -E_{\mathbf{y}_k}\left[\frac{\partial^2 \Lambda(\mathbf{y}_k|\mathbf{\Theta}^k)}{\partial {\sigma^2}^2}\right], k = 1, 2, ..., L, \\ \\ -E_{\mathbf{y}_k}\left[\frac{\partial^2 \Lambda(\mathbf{y}_k|\mathbf{\Theta}^k)}{\partial {\sigma_k^2}^2}\right], k = L+1, L+2, ..., K, \end{cases} \tag{25}$$

$$p = \sum_{l=1}^{L} d_l. \tag{26}$$

We hence obtain the following matrix form for the FIM :

$$\mathbf{I}(\theta) = \begin{pmatrix} \mathbf{I}_1 & \mathbf{0}_{(L+1)\times 2(K-L)} \\ \mathbf{0}_{2(K-L)\times(L+1)} & \mathbf{I}_2 \end{pmatrix}, \tag{27}$$

with

$$\mathbf{I}_1 = \begin{pmatrix} b_1 & 0 & 0 & \cdots & 0 & c_1 \\ 0 & b_2 & 0 & \cdots & 0 & c_2 \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 0 & b_L & c_L \\ c_1 & c_2 & \cdots & \cdots & c_L & p \end{pmatrix}, \tag{28}$$

$$\mathbf{I}_2 = \begin{pmatrix} \mathbf{J}_{L+1}(\theta^{L+1}) & \mathbf{0}_{2\times 2} & \cdots & \mathbf{0}_{2\times 2} \\ \mathbf{0}_{2\times 2} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0}_{2\times 2} \\ \mathbf{0}_{2\times 2} & \cdots & \mathbf{0}_{2\times 2} & \mathbf{J}_K(\theta^K) \end{pmatrix}, \tag{29}$$

the elements of $\mathbf{I}_2$ being:

$$\mathbf{J}_k(\theta^k) = \begin{pmatrix} b_k & c_k \\ c_k & d_k \end{pmatrix}, \quad k = L+1, L+2, ..., K. \tag{30}$$

Finally, the CRLB of the SNR estimates covariance matrix is given by :

$$\mathbf{CRLB}(\rho) = \begin{pmatrix} \mathbf{G}_1\mathbf{I}_1^{-1}\mathbf{G}_1^T & \mathbf{0}_{L\times(K-L)} \\ \mathbf{0}_{(K-L)\times L} & \mathbf{G}_2\mathbf{I}_2^{-1}\mathbf{G}_2^T \end{pmatrix}. \tag{31}$$

Since we are interested in the per-carrier SNR estimation, our aim is to compute the CRLB for each subcarrier, which is given by the diagonal elements of the matrix $\mathbf{CRLB}(\rho)$:

$$CRLB^k(\rho_k) = [\mathbf{CRLB}(\rho)]_{kk}, \quad k = 1, 2, ..., K. \tag{32}$$

It can be seen that, for the $(K-L)$ last subcarriers experiencing different noise powers, we have :

$$\mathbf{G}_2\mathbf{I}_2^{-1}\mathbf{G}_2^T = \text{diag}\left(\left[CRLB_{SC}^{L+1}(\rho_{L+1}), ..., CRLB_{SC}^{K}(\rho_K)\right]\right), \tag{33}$$

where $CRLB_{SC}^k(\rho_k)$, for $k = L+1, L+2, ..., K$, is the CRLB of the SNR estimate over the k^{th} subcarrier that can be achieved with a traditional single-carrier system. Since these CRLBs have already been computed [5], we now focus only on the CRLBs for the first L carriers that experience the same noise power.

The expressions of the diagonal elements of $\mathbf{I}_1^{-1}$ are given in [7], [8]. Hence we obtain the expression of the CRLB on the k^{th} tone, for $k = 1, 2, ..., L$:

$$\begin{aligned} CRLB^k(\rho_k) &= \frac{10}{\ln(10)\rho_k^2}\left[\mathbf{I}_1^{-1}\right]_{kk} &(34)\\ &= \frac{10}{\ln(10)\rho_k^2}\,\frac{\frac{p}{b_k} - \sum_{l=1,l\neq k}^{L}\frac{c_l^2}{b_l b_k}}{p - \sum_{l=1}^{L}\frac{c_l^2}{b_l}}. &(35)\end{aligned}$$

The computation of the coefficients b_l, c_l and d_l , for $l = 1, 2, ..., L$, depends on the estimator's category we are interested in, since the LLF changes as we use a DA SNR estimator or an NDA SNR estimator.

A. Data-Aided SNR Estimation

In this case, the transmitted data is known at the receiver. The LLF on the k^{th} of the first L tones is then given by:

$$\begin{aligned} \Lambda(\mathbf{y}_l|\mathbf{\Theta}^l) &= \ln\left(\Pr\left[\mathbf{y}_l|\mathbf{\Theta}^l, \mathbf{a}_l\right]\right), &(36)\\ &= \sum_{n=1}^{N}\ln\left(\Pr\left[y_l(n)|\mathbf{\Theta}^l, a_l(n)\right]\right), &(37)\end{aligned}$$

where

$$\Pr\left[y_l(n)|\mathbf{\Theta}^l, a_l(n)\right] = \frac{1}{2\pi\sigma^2}\exp\left(\frac{-|y_l(n) - S_l a_l(n)|^2}{2\sigma^2}\right). \tag{38}$$

The coefficients b_l, c_l and d_l , $l = 1, 2, ..., L$ are available in closed-form. Omitting algebraic details for the sake of clarity, we give their final expressions here, for $l = 1, 2, ..., L$:

$$\begin{aligned} b_l &= \frac{N}{2\rho_l}, &(39)\\ c_l &= \frac{N}{2\sigma^2}, &(40)\\ d_l &= \frac{N}{\sigma^4}\left(1 + \frac{\rho_l}{2}\right). &(41)\end{aligned}$$

It can be seen in (35) that the terms in σ^2 disappear in the final expression of the CRLB, which ensures that the CRLB does not depend on σ^2.

B. Non-Data-Aided SNR Estimation

In this case, the LLF on the l^{th} of the first L tones is given by:

$$\begin{aligned} \Lambda(\mathbf{y}_l|\mathbf{\Theta}^l) &= \ln\left(\Pr\left[\mathbf{y}_l|\mathbf{\Theta}^l\right]\right), &(42)\\ &= \sum_{n=1}^{N}\ln\left(\Pr\left[y_l(n)|\mathbf{\Theta}^l\right]\right), &(43)\end{aligned}$$

where

$$\Pr\left[y_l(n)|\mathbf{\Theta}^l\right] = \frac{1}{M}\sum_{m=1}^{M}\frac{1}{2\pi\sigma^2}\exp\left(\frac{-|y_l(n) - S_l c_m|^2}{2\sigma^2}\right). \tag{44}$$

The desired coefficients cannot be derived in closed-form in this case for any constellation (only for square QAM constellations [6]), but they can be easily computed numerically for any constellation using a Gauss-Hermitean quadrature [5].

IV. Subcarriers SNR ML Estimators

We derive in this section two ML SNR estimators for DA and NDA modes. In the DA case, we manage to obtain a closed-form expression. Since it is not tractable in the NDA case, we use the Expectation-Maximization (EM) algorithm to find the ML estimates. Moreover, since the obtained ML SNR estimates for the last $K-L$ subcarriers are exactly the ones obtained by a single-carrier ML SNR estimator (see [9]), we focus henceforth on the SNR estimates for the first L subcarriers that share information about the noise power.

A. Data-Aided SNR Estimation

We assume here the transmitted data $\{a_k(n)\}, k = 1, 2, ..., K, n = 1, 2, ..., N$, to be perfectly known at the receiver, and the channel phases to have been perfectly known at the receiver. The global LLF over the first L subcarriers can be written as:

$$\Lambda(\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_L|\rho_1, \rho_2, ..., \rho_L, \sigma^2) = \sum_{l=1}^{L}\sum_{n=1}^{N}\left(-\ln(2\pi\sigma^2) - \frac{|y_l(n) - S_l e^{j\phi_l}a_l(n)|^2}{2\sigma^2}\right). \tag{45}$$

We can easily derive the LLF with respect to S_l , for $l = 1, 2, ..., L$, as:

$$\frac{\partial\Lambda}{\partial S_l} = 2\sum_{n=1}^{N}\left[S_l|a_l(n)|^2 - \Re(y_l(n)e^{-j\phi_l}a_l^*(n))\right], \tag{46}$$

and σ^2, as:

$$\frac{\partial\Lambda}{\partial\sigma^2} = \frac{-NL}{\sigma^2} + \frac{1}{2\sigma^4}\sum_{l=1}^{L}\sum_{n=1}^{N}|y_l(n) - S_l e^{j\phi_l}a_l(n)|^2. \tag{47}$$

We then easily obtain the ML parameter estimates by nulling these derivatives:

$$S_l = \frac{\sum_{n=1}^{N} \Re(y_l(n)a_l^*(n))}{\sum_{n=1}^{N} |a_l(n)|^2}, \text{for } l = 1, 2, ..., L, \quad (48)$$

$$\sigma^2 = \frac{\sum_{l=1}^{L}\sum_{n=1}^{N} |y_l(n) - S_l a_l(n)|^2}{2NL}. \quad (49)$$

B. Non-Data-Aided EM-based SNR Estimation

The following ML NDA SNR estimator is based on the EM algorithm [10]. Roughly speaking, the EM algorithm can be described as follows. It runs in two steps at each iteration:

$$\mathrm{E-Step}: Q\left(\mathbf{\Theta}^{(i)}, \mathbf{\Theta}^{(i-1)}\right) = E\left[\mathcal{L}\left(\mathbf{\Theta}^{(i)}|\mathbf{Y}, \mathbf{A}|\mathbf{\Theta}^{(i-1)}, \mathbf{Y}\right)\right],$$

$$\mathrm{M-Step}: \mathbf{\Theta}^{(i)} = \arg\max Q\left(\mathbf{\Theta}^{(i)}, \mathbf{\Theta}^{(i-1)}\right),$$

where :

$$\mathcal{L}\left(\mathbf{\Theta}^{(i)}|\mathbf{Y}, \mathbf{A}\right) = \ln\left(\Pr\left[\mathbf{Y}, \mathbf{A}|\mathbf{\Theta}^{(i)}\right]\right). \quad (50)$$

At the i^{th} iteration, the algorithm computes $Q\left(\mathbf{\Theta}^{(i)}, \mathbf{\Theta}^{(i-1)}\right)$ during the E-Step and finds the parameter $\mathbf{\Theta}^{(i)}$ that maximizes it during the M-Step. The M-Step requires the knowledge of the introduced phase distortions $\{\phi_l\}_{l=1,2,...,L}$. However, in the following, we assume these phases to be unknown at the receiver, and we estimate them at each iteration. The parameter vector then becomes:

$$\mathbf{\Theta} = \left[S_1, S_2, ..., S_L, \sigma^2, \phi_1, \phi_2, ..., \phi_L\right]. \quad (51)$$

In fact, the estimates obtained by estimating the channel phases with the EM algorithm are better at low SNR values (SNR $<$ 2 dB). Indeed, the LLF is very flat around its maximum with respect to $\{S_l\}_{l=1,2,...,L}$ and σ^2 in this SNR region, and it cannot be well approximated with respect to these coordinates. Luckily, the LLF is more abrupt around its maximum with respect to $\{\phi_l\}_{l=1,2,...,L}$ in the same SNR region. The maximum of the LLF is then better approximated when the channel phases are taken as coordinates for the research of the maximum of the LLF. Nevertheless, the following derivation can be easily adapted to the case where the channel phases are perfectly known at the receiver, by simply replacing, at the i^{th} iteration, the phase estimates $\left\{\phi_l^{(i)}\right\}_{l=1,2,...,L}$ by their true values $\{\phi_l\}_{l=1,2,...,L}$.

Using the notations adopted in [9] in the single-carrier case, we can write the likelihood function on the l^{th} subcarrier as follows :

$$L\left(\mathbf{\Theta_l}|\mathbf{y_l}, \mathbf{a_l}\right) = \prod_{n=1}^{N}\prod_{m=1}^{M}\left(\Pr\left[a_l(n) = c_m\right] \times \Pr\left[y_l(n)|a_l(n) = c_m, \mathbf{\Theta}_l\right]\right)^{x_{n,m}(l)}, \quad (52)$$

where $x_{n,m}(l)$ is the l^{th} subcarrier indicator that equals 1 if $a_l(n) = c_m$ and 0 otherwise for $l = 1, 2, ..., K, n = 1, 2, ..., N, m = 1, 2, ..., M$, and $\mathbf{\Theta}_l$ is defined by:

$$\mathbf{\Theta}_l = \left[S_l, \sigma^2, \phi_l\right]. \quad (53)$$

Hence, the global log-likelihood function is given by :

$$\mathcal{L}\left(\mathbf{\Theta}|\mathbf{Y}, \mathbf{A}\right) = \sum_{l=1}^{L}\mathcal{L}\left(\mathbf{\Theta}_l|\mathbf{y}_l, \mathbf{a}_l\right). \quad (54)$$

It follows that :

$$Q\left(\mathbf{\Theta}^{(i)}, \mathbf{\Theta}^{(i-1)}\right) = \sum_{l=1}^{L} Q_l\left(\mathbf{\Theta}_l^{(i)}, \mathbf{\Theta}_l^{(i-1)}\right), \quad (55)$$

and from (52), we have:

$$Q_l\left(\mathbf{\Theta}_l^{(i)}, \mathbf{\Theta}_l^{(i-1)}\right) = \sum_{n=1}^{N}\sum_{m=1}^{M}\left\{x_{n,m}^{(i)}(l)\left(\ln(\sigma^2) + \frac{|y_l(n) - S_l e^{j\phi_l} c_m|^2}{2\sigma^2}\right)\right\}, \quad (56)$$

with:

$$x_{n,m}^{(i)}(l) = E\left[x_{n,m}(l)|y_l(n), \mathbf{\Theta}_l^{(i-1)}\right] = \frac{\exp\left\{\frac{-\left|y_l(n) - S_l^{(i-1)} e^{j\phi_l^{(i-1)}} c_m\right|^2}{2\sigma^{2(i-1)}}\right\}}{\sum_{p=1}^{M}\exp\left\{\frac{-\left|y_l(n) - S_l^{(i-1)} e^{j\phi_l^{(i-1)}} c_p\right|^2}{2\sigma^{2(i-1)}}\right\}}. \quad (57)$$

The components of $\mathbf{\Theta}$ are independent and its maximum is then simply given by differentiating Q with respect to each component of $\mathbf{\Theta}$ and setting each partial derivative to zero:

$$\frac{\partial Q}{\partial \phi_l} = \sum_{n=1}^{N}\sum_{m=1}^{M}\frac{-jS_l x_{n,m}^{(i)}(l)}{2\sigma^2}\left[e^{j\phi_l} c_m y_l(n)^* - e^{-j\phi_l} c_m^* y_l(n)\right],$$

$$\frac{\partial Q}{\partial \phi_l} = 0 \Longrightarrow \phi_l^{(i)} = \frac{1}{2}\arg\left(\frac{\sum_{n=1}^{N}\sum_{m=1}^{M} x_{n,m}^{(i)}(l) y_l(n)}{\sum_{n=1}^{N}\sum_{m=1}^{M} x_{n,m}^{(i)}(l) y_l(n)^*}\right).$$

For $l = 1, 2, ..., L$, we finally have :

$$\phi_l^{(i)} = \arg\left(\sum_{n=1}^{N}\sum_{m=1}^{M} x_{n,m}^{(i)}(l) y_l(n)\right), \quad (58)$$

$$S_l^{(i)} = \frac{\sum_{n=1}^{N}\sum_{m=1}^{M} x_{n,m}^{(i)}(l)\left(c_m y_l(n)^* e^{j\phi_l^{(i)}} + c_m^* y_l(n) e^{-j\phi_l^{(i)}}\right)}{2 \times \sum_{n=1}^{N}\sum_{m=1}^{M} x_{n,m}^{(i)}(l)|c_m|^2}, \quad (59)$$

and:

$$\sigma^{2(i)} = \frac{\sum_{l=1}^{L}\sum_{n=1}^{N}\sum_{m=1}^{M} x_{n,m}^{(i)}(l)\left|y_k(n) - S_l^{(i)} e^{j\phi_l^{(i)}} c_m\right|^2}{2LN}. \quad (60)$$

It has to be mentioned also that this estimator does not provide accurate phase estimates $\left\{\widehat{\phi}_k\right\}_{k=1,2,...,K}$. However, the LLF is actually periodic with respect to each channel phase $\phi_l, l = 1, 2, ..., L$, its period being equal to the smallest angle leaving the constellation invariant by rotation. The ambiguity on the estimation of the phases is consequently reduced and the SNR estimates $\{\widehat{\rho}_k\}_{k=1,2,...,K}$ are still sufficiently accurate.

V. Simulation Results

We include in this section some graphical representations of the performances of our estimators and the corresponding CRLBs, for different values of L. We plot the mean square error (MSE) on the first of the L tones that share the same noise power. The case $L = 1$ corresponds to the MSE over each of the remaining $K - L$ tones, since the SNR estimation on these last subcarriers is exactly the same than the one obtained with a traditional single-carrier system. All the simulations were run over 2000 Monte-Carlo simulations.

Fig. 1 presents the MSE for different constellations (QPSK, 8-PSK and 32-QAM) and their corresponding CRLBs for $L = 4$. We can see that the MSE of our NDA ML estimator reaches the CRLB for each of the considered constellations, and for only 4 subcarriers experiencing the same noise power. In the DA case, the ML estimation and the corresponding CRLB do not depend on the constellation. The obtained DA MSE reaches as well the CRLB. Moreover, the NDA CRLBs converge to the DA CRLB at high SNR, and the convergence speed depends on the constellation size.

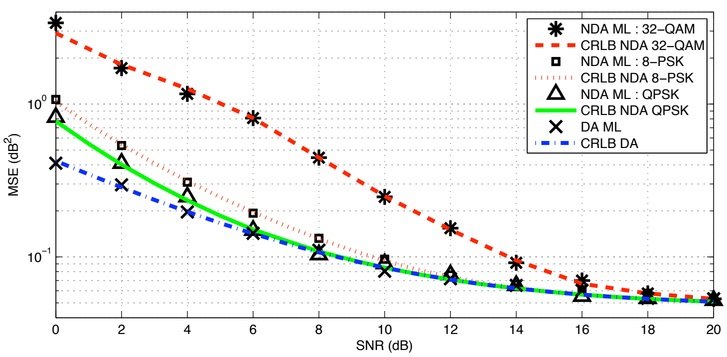

Fig. 1. ML estimators performances comparison. $L = 4, N = 100$.

In Fig. 2, we compare the performances of our NDA ML estimator for different values of L. We clearly see that estimation accuracy increases with L. This is due to the fact that our estimator exploits the cross information shared by the first L subcarriers.

VI. Conclusion

In this paper, we derived analytical expressions for the CRLBs of the per-carrier SNR estimates as a function of different coefficients. We developed a closed-form expression for these coefficients in the DA case, and an easy way to compute them numerically in the NDA case. We also derived two ML subcarrier SNR estimators in both cases. We derived closed-form expressions for SNR estimates in the DA case, and implemented the EM algorithm to obtain the ML per-carrier SNR estimates in the NDA case. We showed that the two estimators reach their CRLBs over a wide range of SNR values. We also confirmed that exploiting the cross information between the subcarriers increases the performances of the estimators significantly.

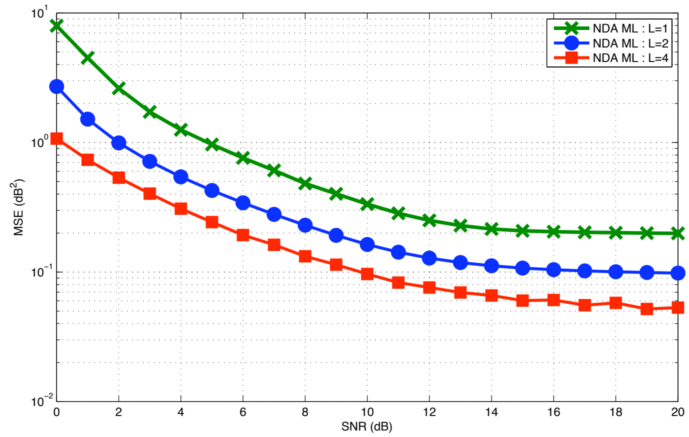

Fig. 2. NDA ML estimator performances comparison with different values of L. $N = 100$, 8-PSK.

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