This full text paper was peer reviewed at the direction of IEEE Communications Society subject matter experts for publication in the WCNC 2007 proceedings.

Generalized Moment-Based Method for SNR Estimation

Mohammed Bakkali[1], Alex Stéphenne[1,2] and Sofiène Affes[1]

[1]INRS-EMT, 800, de la Gauchetière, Bureau 6900, Montreal, H5A 1K6.
[2] Ericsson Canada, 8400, Decarie Blvd, Montreal, H4P 2N2.
Emails:{bakkali,affes}@emt.inrs.ca, alex.stephenne@ericsson.com.

***Abstract*— This paper investigates non-data-aided (NDA) SNR estimation for QAM transmission over additive white Gaussian noise channels. It proposes a novel class of moment-based SNR estimators. This class is found to be a generalization of the well-known moment-based M_2M_4 SNR estimation method for PSK modulation. The performance of the proposed estimators is evaluated for M-PSK and rectangular 16-QAM modulation.**

I. Introduction

In wireless communication systems, the baseband received signal can often be modeled as the superposition of attenuated and phase shifted transmitted signal and additive noise. The signal-to-noise ratio (SNR) is used to describe the relative contributions of the true signal and the background noise. The SNR is one of the important measures of the reliability of the received data and of the channel quality. Performance of various digital communication applications, such as power control, bit error estimation, and turbo decoding, depends on the knowledge of the signal-to-noise ratio.

SNR estimation for AWGN channels has been studied in several works [1]-[4]. Generally speaking, these estimators can be divided into two classes. One class is for data-aided estimators which assume the knowledge of the transmitted data, or that the transmitted data can be reconstructed from the received data and used by the estimator as if it was perfectly reconstructed. The other class is for non-data-aided estimators. For this class of estimators, the transmitted data remain unknown to the receiver. Although the pilot data-aided estimators perform better, the main advantage of the NDA estimators is their bandwidth efficiency due to the elimination of training sets.

Over the past years, there has been growing interest in the use of non-constant modulus constellations. Unfortunately, the topic of SNR estimation for this type of modulation is rarely discussed in the literature. In [1], as in many other studies, only the case of PSK modulation is addressed. However, in [5] the authors present a class of estimators for QAM constellations.

In this paper, a novel class of SNR estimators is presented for QAM modulation over AWGN complex channels. The new method is a NDA moment-based method. This class is found to be a generalization of the M_2M_4 estimator, a very well known moment-based SNR estimator for M-PSK modulation [2].

The study of this class establishes that the M_2M_4 is not always the best estimator to use within this class. Simulation results demonstrate that one specific instance of the estimators class clearly performs better than the M_2M_4 estimator for PSK and remains comparable in performance for 16-QAM.

II. System Model and Novel Moment-Based NDA Estimators

A. System Model

We consider QAM modulation over a frequency-flat fading channel. Symbol-spaced samples at the matched filter output are given by

$$y_n = ha_n + w_n, \tag{1}$$

where $n = 1, 2, ...N$ is the time index in the observation interval, y_n is the received signal, $h = Ae^{j\phi}$ is the channel coefficient assumed complex and constant over the observation interval, a_n is the transmitted QAM symbol and w_n is a realization of a zero mean complex white Gaussian random process of variance $N_0 = 2\sigma^2$. The SNR of the received symbol is given by:

$$\rho = \frac{A^2}{2\sigma^2} = \frac{S}{N_0}. \tag{2}$$

We assume that a_n comes from a constellation that has Q different amplitudes $A_1, A_2..., A_Q$, with probabilities of $p_1, p_2, ..., p_Q$, respectively. Without loss of generality, we assume that the average power of the constellation to be normalized to 1. With these assumptions, it was shown in [5] that the probability density function (PDF) for $|y_n|$ is a mixed Ricean distribution expressed as follows:

$$f_{|y_i|}(y) = \sum_{i=1}^{Q} p_i \frac{y}{\sigma^2} \exp\left(-\rho_i - \frac{y^2}{2\sigma^2}\right) \mathrm{I}_0\left(y\sqrt{\frac{2\rho_i}{\sigma^2}}\right), \tag{3}$$

where $\rho_i = \rho A_i^2$ and $\mathrm{I}_0(.)$ is the Bessel function of the first kind and of order 0. Let us define the kth moment of $|y_n|$ as:

$$M_k = \mathrm{E}\left\{|y_n|^k\right\}. \tag{4}$$

For PSK modulation, M_k is the kth moment of a Ricean variable:

$$M_k = (2\sigma)^{k/2}\,\Gamma\left(k/2+1\right)\exp\left(-\rho\right){}_1F_1\left(k/2+1;1;\rho\right), \tag{5}$$

where ${}_1F_1\left(.;.;.\right)$ is the confluent hypergeometric function and Γ is the gamma function.

For QAM modulation, M_k is the kth moment of a mixed Ricean distribution:

$$\begin{aligned} M_k &= \sum_{i=1}^{Q} p_i\,(2\sigma)^{k/2}\,\Gamma\left(1+k/2\right)\exp\left(-\rho A_i^2\right) \\ &\quad \times {}_1F_1\left(k/2+1;1;\rho A_i^2\right). \end{aligned} \tag{6}$$

For the sake of demonstration, we will present our algorithm derivation for PSK modulation before looking at QAM signals.

B. PSK Signals

When a_n comes from a PSK signal, the M_k in (5) is expressed in terms of the confluent hypergeometric function ${}_1F_1\left(.;.;.\right)$. A very interesting property of this function is conveyed by the following recurrence identities:

$$\begin{aligned} (b-a)\,{}_1F_1\left(a-1,b,z\right) &+ (2a-b+z)\,{}_1F_1\left(a,b,z\right) \\ &- a\,{}_1F_1\left(a+1,b,z\right) = 0, \end{aligned} \tag{7}$$

where a, b and z are real numbers. It is also interesting to express the successive moments of order k, $k+2$ and $k+4$:

$$\begin{aligned} M_k &= (2\sigma)^{k/2}\,\Gamma\left(k/2+1\right)e^{-\rho}\;{}_1F_1\left(k/2+1;1;\rho\right), \\ M_{k+2} &= (2\sigma)^{k/2+1}\,\Gamma\left(k/2+2\right)e^{-\rho}\;{}_1F_1\left(k/2+2;1;\rho\right), \\ M_{k+4} &= (2\sigma)^{k/2+2}\,\Gamma\left(k/2+3\right)e^{-\rho}\;{}_1F_1\left(k/2+3;1;\rho\right). \end{aligned} \tag{8}$$

Using (7), with $a = k/2+2$, $b = 1$ and $z = \rho$, we can write the following expression:

$$\begin{aligned} -\left(1+\frac{k}{2}\right){}_1F_1\left(\frac{k}{2}+1,1,\rho\right) &+ (k+3+\rho)\,{}_1F_1\left(\frac{k}{2}+2,1,\rho\right) \\ &- \left(\frac{k}{2}+2\right){}_1F_1\left(\frac{k}{2}+3,1,\rho\right) = 0. \end{aligned} \tag{9}$$

Combining (9) and (8), we have:

$$\begin{aligned} -\left(1+\frac{k}{2}\right)\frac{M_k}{N_0^{\frac{k}{2}}\Gamma\left(\frac{k}{2}+1\right)} &+ (k+3+\rho)\frac{M_{k+2}}{N_0^{\frac{k}{2}+1}\Gamma\left(\frac{k}{2}+2\right)} \\ &- \left(\frac{k}{2}+2\right)\frac{M_{k+4}}{N_0^{\frac{k}{2}+2}\Gamma\left(\frac{k}{2}+3\right)} = 0. \end{aligned} \tag{10}$$

This equation simplifies to:

$$M_{k+4} = -\left(1+\frac{k}{2}\right)^2 M_k N_0^2 + (k+3+\rho)\,M_{k+2}N_0. \tag{11}$$

In order to eliminate ρ, we use the second envelope moment:

$$M_2 = S + N_0 = (\rho+1)\,N_0. \tag{12}$$

Finally, we have:

$$M_{k+4} = -\left(1+\frac{k}{2}\right)^2 M_k N_0^2 + (k+2)\,M_{k+2}N_0 + M_2 M_{k+2}. \tag{13}$$

This equation allows the estimation of the noise variance from estimates of the appropriate moments obtained previously via sample averaging with:

$$M_k \approx \frac{1}{N}\sum_{n=1}^{N}|y_n|^k. \tag{14}$$

The estimation of the SNR is obtained using $\hat{\rho} = \frac{M_2 - N_0}{N_0}$. For the special case $k = 0$, we find that equation (13) is equivalent to:

$$M_4 = -N_0^2 + 2M_2N_0 + M_2^2 \tag{15}$$

which amounts to the M_2M_4 estimator. Indeed, the M_2M_4 estimator is based on the following equation [1]:

$$M_4 = k_a S^2 + 4SN_0 + k_w N_0^2, \tag{16}$$

where $k_a = \mathrm{E}\{|a_n|^4\}/\mathrm{E}\{|a_n|^2\}^2$ and $k_w = \mathrm{E}\{|w_n|^4\}/\mathrm{E}\{|w_n|^2\}^2$ are the kurtosis of the complex signal and the complex noise. Indeed, for a PSK signal over complex AWGN channels, we have $k_a = 1$ and $k_w = 2$. Hence, (16) simplifies to:

$$M_4 = S^2 + 4SN_0 + 2N_0^2, \tag{17}$$

expressed also by using (12):

$$M_4 = -N_0^2 + 2M_2N_0 + M_2^2. \tag{18}$$

We can demonstrate that equation (13) is valid for all values of $k \geq -2$. Hence, we can derive a class of SNR estimators which relies on this equation. This class is referred to as the generalized moment-based (GM) SNR estimators. The estimator using the k, $k+2$ and $k+4$ order moments is referred to as the GM_k estimator. For example, GM_0 is equivalent to M_2M_4.

C. QAM Signals

A QAM constellation could be seen as the sum of different PSK constellations with different amplitudes A_i. The kth envelope moment for the constellation of amplitude A_i is given by:

$$M_{k,i} = (2\sigma)^{\frac{k}{2}}\,\Gamma\left(1+\frac{k}{2}\right)\exp\left(-\rho A_i^2\right){}_1F_1\left(\frac{k}{2}+1;1;\rho A_i^2\right). \tag{19}$$

The kth envelope moment for the all constellation is given by:

$$M_k = \sum_{i=1}^{Q} p_i M_{k,i}. \tag{20}$$

Equation (11) for each PSK constellation yields to:

$$M_{k+4,i} = -\left(1+\frac{k}{2}\right)^2 M_{k,i}N_0^2 + \left(k+3+\rho A_i^2\right)M_{k+2,i}N_0. \tag{21}$$

Combining (20) and (21), we have:

$$\begin{aligned} M_{k+4} &= -\left(1+\frac{k}{2}\right)^2 M_k N_0^2 + (k+3)\, M_{k+2} N_0 \\ &\quad + (M_2 - N_0) \times \sum_{i=1}^{Q} p_i A_i^2 M_{k+2,i}, \end{aligned} \tag{22}$$

which can be expressed as:

$$\begin{aligned} M_{k+4} &= -\left(1+\frac{k}{2}\right)^2 M_k N_0^2 + (k+3)\, M_{k+2} N_0 \\ &\quad + (M_2 - N_0) \times \mathrm{E}\{|a_n|^2 |y_n|^{k+2}\}. \end{aligned} \tag{23}$$

Substituting $k = 0$ in (23), we have:

$$M_4 = -N_0^2 + 3M_2N_0 + (M_2 - N_0)\,\mathrm{E}\{|a_n|^2|y_n|^2\}. \tag{24}$$

On the other hand:

$$\begin{aligned} \mathrm{E}\{|a_n|^2|y_n|^2\} &= \mathrm{E}\{(a_n y_n) * (a_n y_n)^*\} \\ &= \mathrm{E}\{(a_n (h a_n + w_n)) * (a_n (h a_n + w_n))^*\} \\ &= \mathrm{E}\{a_n^2\}\,(k_a S + N_o)\,. \end{aligned} \tag{25}$$

Combining (24) and (25), we have the following equation:

$$M_4 = (k_a - 2)\, N_0^2 + (4 - 2k_a)\, M_2 N_0 + k_a M_2^2, \tag{26}$$

which we use in GM_0 to estimate N_0. As in the PSK case, GM_0 is a version of the M_2M_4 estimator applied to QAM signals.

It turns out that, for k even, the term $\mathrm{E}\{|a_n|^2|y_n|^{k+2}\}$ is a polynomial function of N_0 of order $\frac{k+4}{2}$. Hence, in this case, equation (23) is a polynomial equation of order $\frac{k+4}{2}$. For example, for a 16-QAM signal with $k = 2$, N_0 satisfies this equation:

$$\begin{aligned} -1.32N_0^3 - 3.96M_2N_0^2 &+ \left(5M_4 - 0.6M_2^2\right) N_0 \\ &+ 1.96M_2^3 - M_6 = 0. \end{aligned} \tag{27}$$

Equation (23) is more difficult to handle in the case of k odd. For example, for $k = -1$, we have the following equation:

$$M_3 = -\left(1/2\right)^2 M_{-1}N_0^2 + 2M_1N_0 + (M_2 - N_0)\,\mathrm{E}\{|a_n|^2|y_n|\}, \tag{28}$$

with

$$\begin{aligned} \mathrm{E}\{|a_n|^2|y_n|\} &= \sum_{i=1}^{Q} p_i A_i^2\, (2\sigma)^{1/2}\, \Gamma\left(3/2\right) \exp\left(-\rho A_i^2\right) \\ &\quad \times {}_1F_1\left(3/2; 1; \rho A_i^2\right). \end{aligned} \tag{29}$$

The problem amounts to resolving an equation of the type $\mathrm{f}(\rho) = 0$. For $k = -1$, f is found to be monotonic. Hence, we can simply use the dichotomy method to resolve this equation (See Fig. 1). Finally, rederiving (23) without the assumption $\sum_{Q}^{i=1} p_i A_i^2 = 1$, we find the general expression (30) which is applicable to QAM signals over AWGN channels.

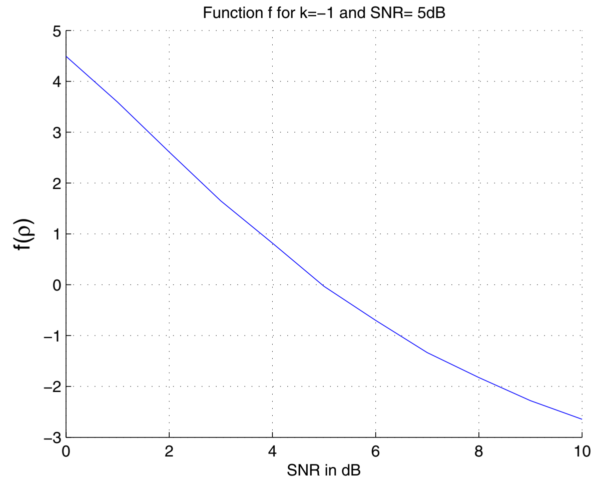

Fig. 1. Function f for $k = -1$ with 16-QAM signals and $SNR = 5dB$.

III. Simulation Results and Comparison

Simulation results are provided for the novel class of SNR estimators GM_k. Specifically, 8-PSK and 16-QAM modulated signals over complex AWGN channels are simulated. Notice that since all the estimators are envelope-based estimators, the results for PSK modulation are independent from the constellation order. For comparison, we provide also the performance of M_2M_4 (GM_0) and the best estimator introduced in [5] (i.e. $\hat{\rho}_{1,2}$ for 8-PSK and $\hat{\rho}_{2,4}$ for 16-QAM). For 8-PSK, we choose the number of symbols $N = 64$. For 16-QAM, $N = 1000$ is used. For the $\hat{\rho}_{1,2}$, a lookup table is needed. We used a grid with spacing of 0.1 corresponding to a table size 100. The DA CRB is also included as a reference [1].

First of all, we start by looking at the performance of the different estimators GM_k for 8-PSK signals. In Fig. 2, we present the NMSE (Normalized Mean Squared Error) for different values of k at different SNR values. It is immediately apparent from Fig. 2 that the NMSE of GM_{-1} is the smallest over the entire range of tested SNR values and outperforms $M_2M_4 \equiv GM_0$, the most popular moment-based estimator. However, the M_2M_4 estimator is better than GM_1.

In Fig. 3, we plot the normalized bias of the estimators under study as a function of the SNR values. In terms of bias, it is observed that GM_{-1} has the smallest bias and hence emerges as the best estimator from the introduced class of estimators. Simulation results not shown here demonstrate that $k = -1$ is optimal in terms of NMSE and bias.

In Fig. 4, with $N = 64$, we compare the NMSE of the estimator GM_{-1}, $\hat{\rho}_{1,2}$, and M_2M_4 for 8-PSK constellation. $\hat{\rho}_{1,2}$ has the lowest NMSE and outperforms M_2M_4 and GM_{-1}. $\hat{\rho}_{1,2}$ employs an equation based on measured $(\frac{M_1^2}{M_2})$ to estimate the SNR using an equation of the form $\hat{\rho}_{1,2} = f_{1,2}^{-1}(\frac{M_1^2}{M_2})$. A disadvantage of this approach is the use of a lookup table. The newly introduced estimator GM_{-1} performs relatively well in comparison with $\hat{\rho}_{1,2}$ and with a similar order of complexity.

Fig. 5 compares the bias for the different estimators. $\hat{\rho}_{1,2}$

$$M_{k+4} = -(1+k/2)^2 M_k N_0^2 + (k+3)\, M_{k+2} N_0 + \frac{M_2 - N_0}{\mathrm{E}\{|a_n|^2\}} \mathrm{E}\{|a_n|^2 |y_n|^{k+2}\}. \tag{30}$$

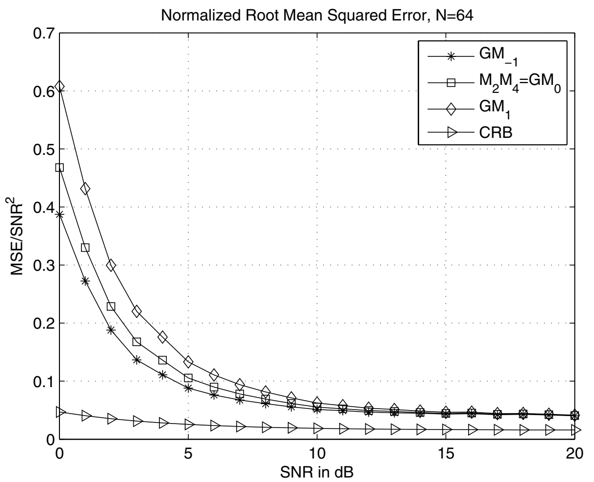

Fig. 2. True SNR normalized mean squared error of different GM_k estimators with 8-PSK signals, $N = 64$.

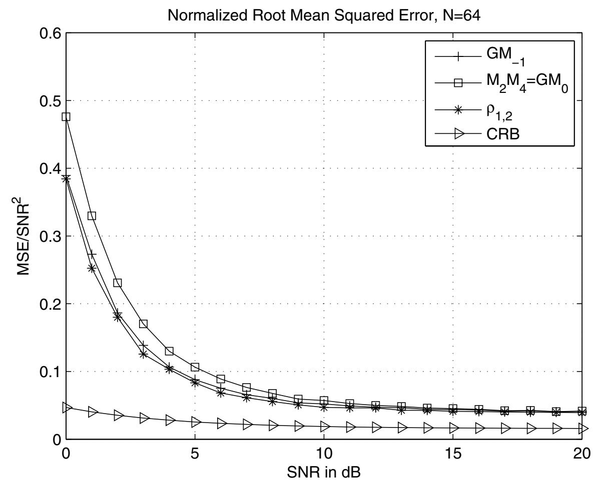

Fig. 4. True SNR normalized mean squared error of the estimators with 8-PSK signal, $N = 64$.

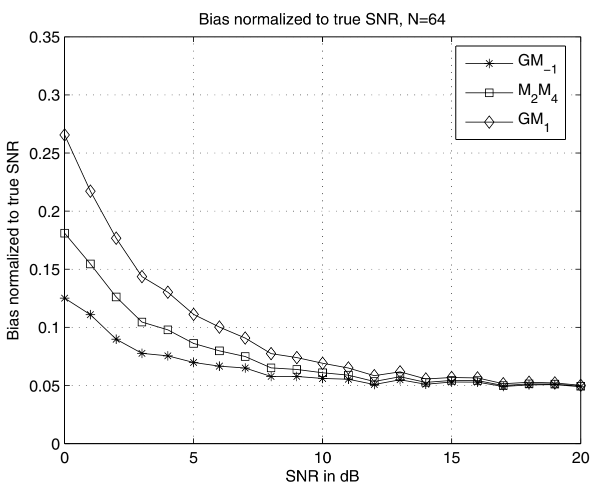

Fig. 3. Normalized bias of different GM_k estimators with 8-PSK signals, $N = 64$.

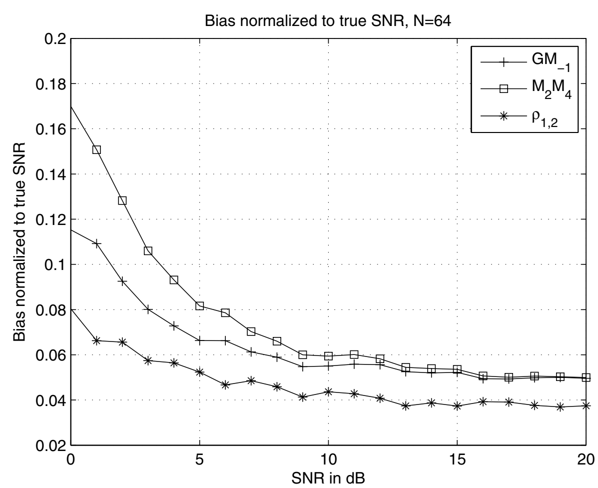

Fig. 5. Normalized bias of the estimators with 8-PSK signals, $N = 64$.

has the lowest bias. At high SNR, when GM_{-1} and M_2M_4 exhibit the same bias, $\hat{\rho}_{1,2}$'s bias is slightly smaller.

Fig. 6 shows the NMSE for $N = 1000$ samples with 16-QAM signals. Further investigation for $\hat{\rho}_{2,4}$ demonstrates that this estimator is equivalent to M_2M_4 in the case of 16-QAM modulation. For GM_{-1}, in order to resolve equation (28), we use a dichotomy algorithm with a precision of 10^{-4}. GM_{-1} offers comparable performance to the M_2M_4 estimator. Nevertheless, M_2M_4 exhibits the same performance without the complexity introduced in the GM_{-1} estimator. Hence, for 16-QAM, among all the GM estimators, GM_0, i.e., M_2M_4 (or $\hat{\rho}_{2,4}$) remains the best estimator. For SNR$>$ 10 dB, the performance degrades gradually with SNR increasing. This effect is common for moment-based methods for QAM signals. At a high SNR range, we have to use longer observation data.

Fig.7 shows the normalized bias vs. the actual SNR. For 16-QAM signals, M_2M_4 is the best estimator in terms of bias. Notice that GM_2 has a large bias while GM_{-1} practically presents the same bias as M_2M_4.

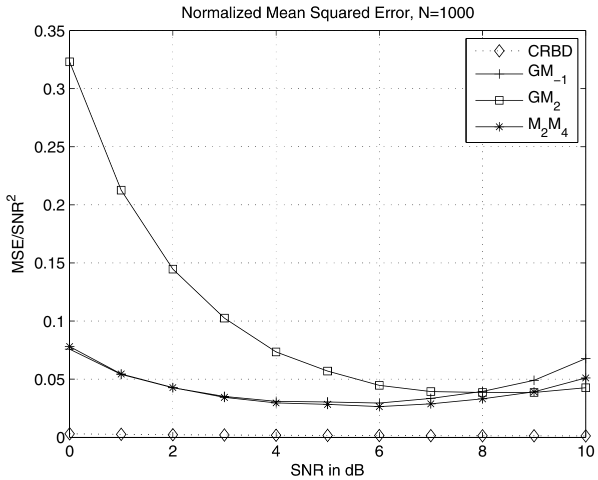

Fig. 6. True SNR normalized mean squared error of the estimators with 16-QAM signals, $N = 1000$.

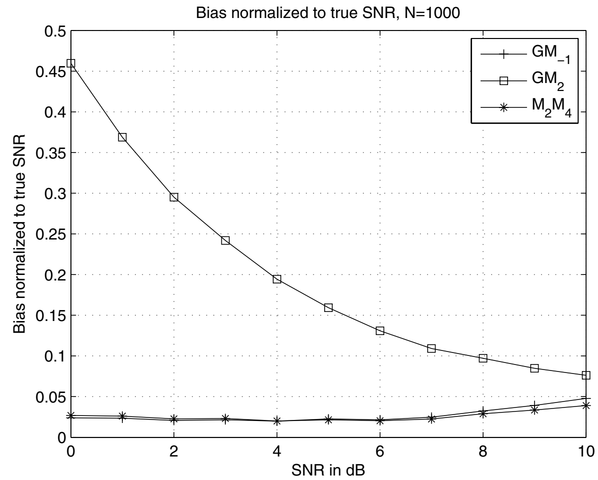

Fig. 7. Normalized bias of the estimators with 16-QAM signals, $N = 1000$.

IV. Conclusion

In this paper we have derived a family of moment-based SNR estimators. It was shown that this class is a generalization of M_2M_4. Simulation results show that for PSK modulation, a more accurate estimator could be derived from this family that outperforms the M_2M_4, considered until this work as the best moment-based estimator both in performance and simplicity. On the other hand, for QAM modulation, M_2M_4 still remains the best candidate in terms of performance and complexity.

References

[1] D.R. Pauluzzi and N.C. Beaulieu,"A Comparison of SNR estimation techniques for the AWGN channel," *IEEE Trans. Comm.*, vol. 48, no. 10, pp. 1681-1691, Oct. 2000.

[2] R. Matzner and F. Engleberger, "An SNR estimation algorithm using fourth-order moments," *in Proc. IEEE Int. Symp. On Information Theory*, Trondheim, Norway, p. 119, June 1994.

[3] C.E. Gilchriest, "Signal-to-noise monitoring," *JPL Space Programs Summary*, vol. IV, no. 37-27, pp. 169-184, June 1966.

[4] R. M. Gagliardi and C. M. Thomas, "PCM data reliability monitoring through estimation of signal-to-noise ratio," *IEEE Trans. Comm.*, vol. COM-16, no. 3, pp. 479-486, June 1968.

[5] P. Gao and C. Tepedelenlioglu, "SNR estimation for nonconstant modulus constellations," *IEEE Trans. Signal Processing,* vol. 53, no.3, pp 865-870, Mar. 2005.