A REDUCED-BIAS SNR ESTIMATOR FOR BPSK MODULATION OVER AWGN CHANNELS

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ABSTRACT

In this paper we propose a new reduced-bias SNR estimator for BPSK modulation over AWGN channels derived from the maximum likelihood (ML) approach. This estimator holds for both the blind and pilot-assisted cases. Simulation results demonstrate the superiority of the new SNR estimator over previous estimators at low SNR, where they usually exhibit a high estimation bias. At high SNR, the new estimator remains comparable in accuracy to the best existing techniques.

1. INTRODUCTION

Signal-to-noise ratio (SNR) is an important measure in several communication systems. The SNR measurement can be needed in various applications such as power control, adaptive modulation and coding and cell hand-off. Several SNR estimation techniques have been proposed for AWGN channels. The SNR estimators can be classified as data-aided (DA) and non data-aided (NDA). DA estimators assume the knowledge of the transmitted data or that the transmitted data can be reconstructed from the received data and used by the estimator as if it was perfectly reconstructed. On the other hand, NDA estimators assume that the data remain unknown to the receiver. A DA estimator that makes use of the perfect knowledge of the transmitted sequence is designated by TxDA (i.e., pilot-assisted). A DA estimator that uses an estimate of the transmitted data sequence from receiver decisions is designated by RxDA (i.e., blind). The NDA and RxDA estimators have the advantage of not reducing the bandwidth efficiency of the communication system, contrary to the TxDA estimators, and hence are more interesting. A comparison of many of these estimators was performed in [1]. From all estimators studied there, the ML RxDA [2] and the M_2M_4 [4] estimators emerge as best candidates. ML RxDA is the best estimator at high SNR but is outperformed by M_2M_4 at low SNR due to the bias caused by receiver decision errors.

In this paper we propose a simple method that reduces the bias of the ML RxDA estimator observed at low SNR. The resulting reduced-bias version of the ML RxDA estimator outperforms the M_2M_4 at low SNR and remains as accurate as the classical ML RxDA estimator at high SNR.

Since most practical communication systems use some form of synchronization and/or training sequences that are known to the receiver, we modify our new algorithm so that it can exploit the presence of both pilot and data symbols.

Simulation results demonstrate the superiority of the new SNR estimator over previous estimators at low SNR, where they usually exhibit a high estimation bias. At high SNR, the new estimator remains comparable in accuracy to the best existing techniques.

2. SYSTEM MODEL AND PROBLEM STATEMENT

We consider the same system model as in [1],[2], that defines the received signal as follows:

$$y_i = Aa_i + w_i, \tag{1}$$

where $i = 1, 2, ...N$ is the time index in the observation interval, y_i is the received signal, A is the signal amplitude assumed positive and constant over the observation interval, a_i is the transmitted BPSK signal and w_i is a realization of a zero mean white Gaussian random process of variance σ^2. The SNR of the received symbol is given by:

$$\rho = \frac{A^2}{\sigma^2}. \tag{2}$$

We assume here that the transmitted symbols a_i are deterministic realizations. Hence, the probability density function (PDF) for the received sample at time index n is expressed as follows:

$$f(y_n) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y_n - Aa_n)^2}{2\sigma^2}}. \tag{3}$$

Hence, the probability density function of a received vector $(y_1, y_2, ..., y_N)$ can be expressed as:

$$f_N(y_1, y_2, ..., y_N) = \prod_{i=1}^{N} f(y_i). \tag{4}$$

In [2], Gagliardi and Thomas introduce the ML SNR estimator for BPSK modulated signals over a real AWGN channel. The ML estimator is expressed as:

$$\hat{\rho} = (\widehat{A^2/\sigma^2}) = (\hat{A})^2/\hat{\sigma}^2, \quad (5)$$

where $\hat{A}$ and $\hat{\sigma}^2$ are the solutions of the likelihood equations:

$$\frac{\partial log f_N}{\partial A} = 0,$$
$$\frac{\partial log f_N}{\partial \sigma^2} = 0. \quad (6)$$

The respective solutions are then found to be:

$$\hat{A} = \frac{1}{N}\sum_{i=1}^{N} y_i a_i, \quad (7)$$

$$\hat{\sigma^2} = \frac{1}{N}\sum_{i=1}^{N} y_i^2 - (\hat{A})^2. \quad (8)$$

Hence, if we use an estimate of the transmitted signal $\hat{a}_i$, we have the classical ML RxDA SNR estimator expressed as:

$$\hat{\rho} = \frac{(\frac{1}{N}\sum_{i=1}^{N} y_i \hat{a}_i)^2}{\frac{1}{N}\sum_{i=1}^{N} y_i^2 - (\frac{1}{N}\sum_{i=1}^{N} y_i \hat{a}_i)^2}. \quad (9)$$

Note in this case that $\hat{a}_i$ is $sign(y_i)$ where $sign$ is the signum function. Hence, $\hat{A}$ may be expressed as:

$$\hat{A} = \frac{1}{N}\sum_{i=1}^{N} |y_i|. \quad (10)$$

In [1], it was shown that the ML RxDA estimator performs equally well at high SNR as the ML TxDA, which relies on pilot data. When the SNR is not high enough to reduce receiver errors, the RxDA estimate exhibits a large bias.

While an exact expression for the bias of TxDA SNR estimator was derived in [2], the calculation of the bias for the RxDA SNR estimator remains challenging. To circumvent this difficulty, we will focus only on the bias in the estimate $\hat{A}$. Indeed, from (10), one notices that $\hat{A}$ is a sum of N random variables $|y_i|$. Hence, $\mathrm{E}(\hat{A}) = \mathrm{E}(\frac{1}{N}\sum_{i=1}^{N}|y_i|) = \mathrm{E}(|y_i|)$. Here, $\mathrm{E}\{.\}$ denotes the expectation. As seen in (3), the random variable y_i is Gaussian with mean Aa_i and variance σ^2. Therefore $|y_i|$ is the absolute value of a Gaussian-distributed variable. The pdf of $|y_i|$ is:

$$f_{|y_i|}(y) = \begin{cases} \frac{1}{\sigma}\left(\phi(\frac{y-Aa_i}{\sigma}) + \phi(\frac{y+Aa_i}{\sigma})\right), & y > 0 \\ 0, & y \leq 0 \end{cases} \quad (11)$$

where ϕ is the standard normal probability density function. Given the distribution of $|y_i|$, the expectation is:

$$\mathrm{E}\{\hat{A}\} = \mathrm{E}\{|y_i|\} = A\left(\sqrt{\frac{2}{\pi}}\frac{e^{-\frac{1}{2}\rho}}{\sqrt{\rho}} + \mathrm{erf}(\sqrt{\frac{\rho}{2}})\right), \quad (12)$$

where $\mathrm{erf}\{.\}$ is the error function.

As expected, $\hat{A}$ is a biased estimator of A. From (12), one can note that the bias in $\hat{A}$ tends to zero as ρ increases. On the other hand, at low SNR, the bias may be large enough to significantly distort the SNR estimation. The poor performance of the classical ML RxDA estimator at low SNR is explained by the presence of this bias.

3. REDUCED-BIAS DA ML-BASED ESTIMATOR

In this section, we introduce a new algorithm to compensate the bias of the RxDA estimator introduced in the previous section.

It is interesting to note that $\hat{A}$ is biased by the factor $(\sqrt{\frac{2}{\pi}}\frac{e^{-\frac{1}{2}\rho}}{\sqrt{\rho}} + \mathrm{erf}(\sqrt{\frac{\rho}{2}}))$. To reduce this bias, one could use the estimator $\hat{\rho}$ to evaluate it then compensate for it. Accordingly we calculate the estimates $\hat{A}$ and $\hat{\rho}$ by using the classical RxDA, then reduce the bias by dividing $\hat{A}$ by $(\sqrt{\frac{2}{\pi}}\frac{e^{-\frac{1}{2}\hat{\rho}}}{\sqrt{\hat{\rho}}} + \mathrm{erf}(\sqrt{\frac{\hat{\rho}}{2}}))$, giving a new estimate $\hat{A}_1$:

$$\hat{A}_1 = \frac{\hat{A}}{\left(\sqrt{\frac{2}{\pi}}\frac{e^{-\frac{1}{2}\hat{\rho}}}{\sqrt{\hat{\rho}}} + \mathrm{erf}(\sqrt{\frac{\hat{\rho}}{2}})\right)}. \quad (13)$$

At last, we calculate the new SNR estimator as:

$$\hat{\rho}_1 = \frac{(\hat{A}_1)^2}{\frac{1}{N}\sum_{i=1}^{N} y_i^2 - (\hat{A}_1)^2}. \quad (14)$$

In fact, this method could be implemented iteratively where in each iteration we estimate the SNR and use it to compensate the bias in (12).

While most SNR estimators use either data or pilot symbols, an SNR estimator exploiting jointly pilot and data symbols can be derived.

In [3], a maximum likelihood based SNR estimator for BPSK using both pilot and data symbols was introduced. Using the same model as before, the received signal is expressed as:

$$y_i = Aa_i + w_i, \quad (15)$$

where $i = 1, 2, ..., P+M$ is now the time index in the observation interval, $i = 1, 2, ..., P$ corresponds to pilot symbols and $i = P+1, P+2, ..., P+M$ corresponds to the data symbols.

The approximate ML-based estimator is derived from the received samples (both data and pilot) giving:

$$\hat{\rho} = \frac{\hat{A}^2}{\frac{1}{P+M}\sum_{i=1}^{P+M} y_i^2 - \hat{A}^2}, \quad (16)$$

where $\hat{A} = \frac{1}{P+M}(\sum_{i=1}^{P} y_i a_i + \sum_{i=P+1}^{P+M} |y_i|)$
It can be shown that:

$$\mathrm{E}\{\hat{A}\} = \frac{A}{P+M}\left(P + M\left[\sqrt{\frac{2}{\pi}}\frac{e^{-\frac{1}{2}\rho}}{\sqrt{\rho}} + \mathrm{erf}\left(\sqrt{\frac{\rho}{2}}\right)\right]\right). \tag{17}$$

The expression above is close to the one found in (12). Our reduced-bias ML estimator is calculated using the following algorithm.

Input $y_1, ..., y_{P+M}$
Input I {number of iterations}

Initialization

calculate $A_0 = \frac{1}{P+M}(\sum_{i=1}^{P} y_i a_i + \sum_{i=P+1}^{P+M} |y_i|)$

calculate $\varepsilon^2 = \frac{1}{P+M}\sum_{i=1}^{P+M} y_i^2$

calculate $\hat{\rho}_0 = \frac{A_0^2}{\varepsilon^2 - A_0^2}$

for iteration $\mathbf{k = 1..I}$

$A_k = A_0 / \left(\frac{P}{N+M} + \frac{M}{P+M}\left[\sqrt{\frac{2}{\pi}}\frac{e^{-\frac{1}{2}\hat{\rho}_{k-1}}}{\sqrt{\hat{\rho}_{k-1}}} + \mathrm{erf}\left(\sqrt{\frac{\hat{\rho}_{k-1}}{2}}\right)\right]\right)$

$\hat{\rho}_k = A_k^2/(\varepsilon^2 - A_k^2)$

End

Output $\hat{\rho}_I$

Obviously this algorithm holds without modification for the blind case described earlier, i.e., $P = 0$.

4. SIMULATION RESULTS AND COMPARISONS

In the following section, Monte Carlo computer simulations are performed over 10000 runs to show the performance of the proposed algorithm. For comparison, we provide the performance of M_2M_4, the classical ML RxDA estimator and the approximate maximum likelihood estimator [3] expressed in (16). We will therefore use the classical CRB [5] as a reference.

Note that all ML-based estimators generate estimates that are in fact biased estimates by a factor of $\frac{N-3}{N}$ [1]. A reduced-bias (RB) ML-based SNR estimator can be derived as follows:

$$\hat{\rho}_{ML-RB} = \frac{N-3}{N}\,\hat{\rho}_{ML}. \tag{18}$$

In our simulations, the reduced-bias versions are used for all ML- based estimators.

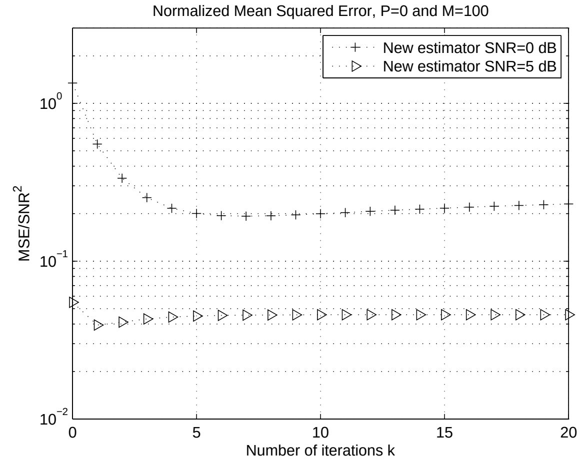

Fig. 1. True SNR normalized mean squared error of the new estimator in the blind case.

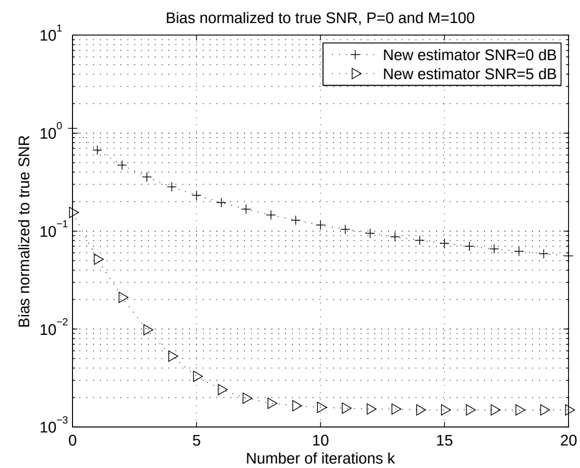

Fig. 2. Normalized bias of the new algorithm in the blind case.

We start by looking at the performance of the new algorithm as a function of the number of iterations. For the sake of simplicity, we only consider for now the blind case ($P = 0$ and $M = 100$). In Fig. 1, we present the normalized mean squared error (NMSE) as a function of the number of iterations at different SNR levels.

At 0 dB, we report a quick saturation in performance improvement beyond the 6^{th} iteration (i.e., $I = 6$). Notice that for each SNR value, we find an optimal number of iterations. As we will see, this is hardly surprising since there is a bias/variance tradeoff requiring an optimal number of iterations for each SNR value. Therefore, in practice, we can change the number of iterations dynamically depending on the range of SNR.

In Fig. 2, we exhibit the normalized bias of the new es-

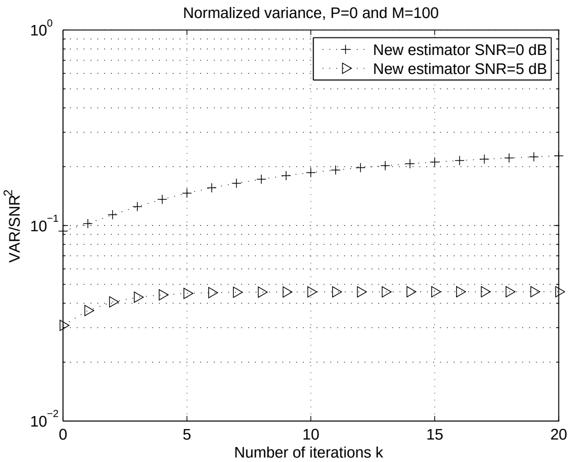

Fig. 3. The normalized variance of the new estimator in the blind case.

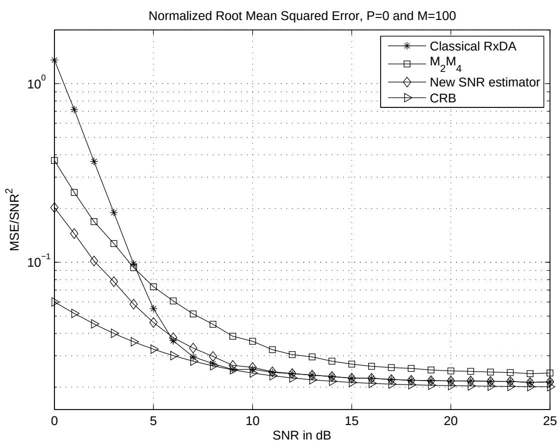

Fig. 4. True SNR normalized mean squared error of the estimators in the blind case with $I = 10$.

timator as a function of the number of iterations. It is shown that the new algorithm converges relatively rapidly. At 0 dB, the bias is reduced by factor 10. This factor is about 40 at 5 dB.

In Fig. 3, we plot the normalized variance for different SNR values versus the number of iterations I. We notice that the new algorithm reduces the bias at the expense of an increase in variance, thereby leading to a bias-variance tradeoff. In fact, the choice of the number of iterations depends on the design criteria involving variance and bias. In many cases the objective may be simply to reduce the bias with a higher number of iterations, less so at higher SNR. Overall, the new algorithm offers flexibility to accommodate various applications with different criteria.

Fig. 4 shows the NMSE as a function of SNR for the estimators in the blind case ($P = 0$ and $M = 100$).

As expected, the classical ML RxDA estimator performs poorly at low SNR (due to errors in symbol decision) but offers an acceptable performance at high SNR reaching the CRB. The M_2M_4 estimator is better than the RxDA estimator at low SNR, but not at hight SNR. The newly proposed estimator outperforms both techniques at low SNR with only $I = 10$ iterations, providing an efficient estimation over a much wider range of SNR.

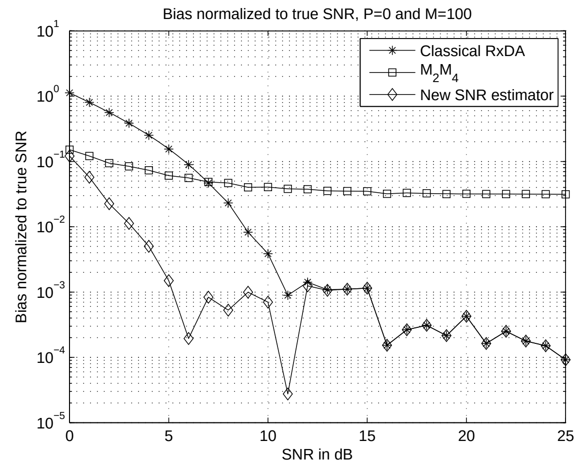

Fig. 5. Normalized bias of the estimators in the blind case with number of iterations for the new estimator ($I = 10$).

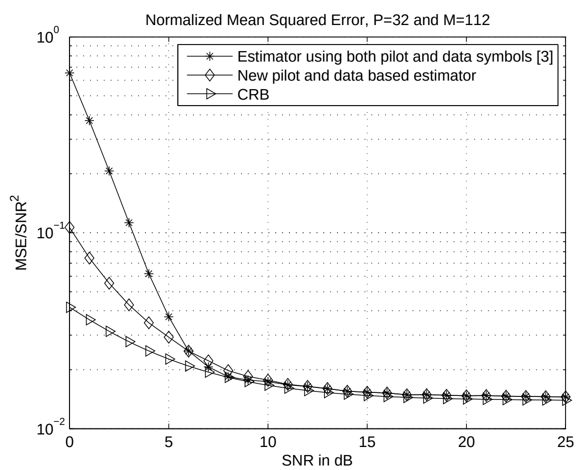

Fig. 6. True SNR normalized mean squared error of the estimators in the pilot-assisted case with number of iterations for the new estimator ($I = 3$).

Fig. 5 compares the bias of the new estimator, the M_2M_4 and the classical RxDA estimators. One sees that our algorithm outperforms the M_2M_4 in terms of bias.

In Fig. 6, we plot the NMSE for the pilot-assisted case ($P = 32$, $M = 112$ and $I = 3$ iterations) compared to the

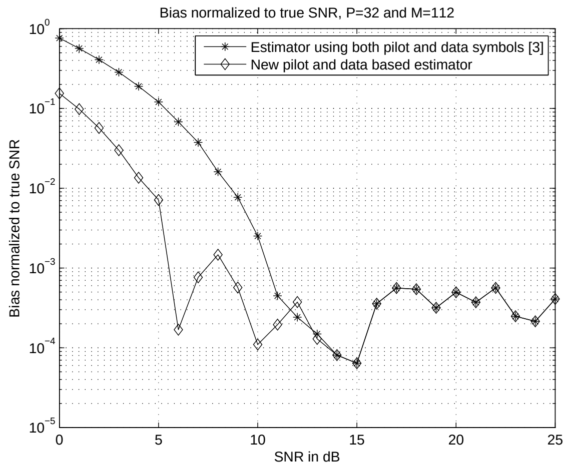

Fig. 7. Normalized bias of the estimators in the pilot-assisted case with number of iterations for the new estimator ($I = 3$).

NMSE of the estimator proposed in [3]. It shows that our estimator outperforms the estimator in [3] over a large range of practical SNR values. The superiority of our algorithm is obvious at low SNR.

Lastly, in Fig. 7 we examine the bias in the pilot-assisted case compared to the bias of the estimator proposed in [3]. Our estimator performs better at low SNR. As an example, for SNR$= 0$ dB, we have a bias gain of about 6 dB.

5. CONCLUSION

A new SNR estimator for BPSK signals is proposed. The algorithm is a data-aided ML-based estimator with bias compensation. Simulations exhibit a performance gain over other previously proposed techniques. Either using data only or combining data and pilot symbols, the new estimator offers efficient performance gain over a wider range of SNR values.

6. REFERENCES

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