A PERFORMANCE COMPARISON BETWEEN SUPERIMPOSED AND TIME-MULTIPLEXED TRAINING – MATHEMATICAL DERIVATION DETAILS

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In this report, we provide the details of the proofs for the asymptotic behaviour of the post-processing SNR for the Time-multiplexed training scheme as well as for the data-dependent superimposed training scheme.

I. TIME-MULTIPLEXING TRAINING SCHEME

In [1], we stated the following theorem:

Theorem 1: Under the asymptotic regime, and conditioned on the channel, the post-processing noise experienced by the i-th antenna at each time j for the TDMT scheme behaves asymptotically as a Gaussian random variable:

$$\mathbb{E}\left[e^{\jmath \Re(s^* \Delta \mathbf{W}_t(i,j))}\right] - e^{-\frac{\sigma_w^2 \delta_t \left[\left(\mathbf{H}^{\mathrm{H}}\mathbf{H}\right)^{-1}\right]_{i,i} |s|^2}{4}} \xrightarrow[N\to\infty]{} 0$$

where

$$\delta_t = c_1(1+r)\frac{\sigma_v^2}{\sigma_P^2} + \frac{\sigma_v^2}{\sigma_w^2} + \frac{c_1(1+r)\sigma_v^4}{\sigma_w^2\sigma_P^2(c_2-1)}.$$

We provide hereafter some elements of the proof.

We recall in [1] that the channel estimate is given by:

$$\widehat{\mathbf{H}}_t = \mathbf{H} + \Delta\mathbf{H}_t$$

where $\Delta\mathbf{H}_t = \mathbf{V}_1\mathbf{P}_t^{\mathrm{H}}\left(\mathbf{P}_t\mathbf{P}_t^{\mathrm{H}}\right)^{-1}$. After applying the zero forcing linear receiver, the effective post-processing noise $\Delta\mathbf{W}_t$ can be written as:

$$\Delta\mathbf{W}_t = -\mathbf{H}^{\#}\Delta\mathbf{H}_t\mathbf{W} + \left(\mathbf{H}^{\#} - \mathbf{H}^{\#}\Delta\mathbf{H}_t\mathbf{H}^{\#}\right)\mathbf{V}_2$$

In the sequel, we propose to determine the asymptotic distribution of the post-processing noise of each entry of the matrix $\Delta\mathbf{W}_t$. Actually the (i,j) entry of $\Delta\mathbf{W}_t$ is given by:

$$\left(\Delta\mathbf{W}_t\right)_{i,j} = -\mathbf{h}_i^{\#}\Delta\mathbf{H}_t\mathbf{w}_j + \left(\mathbf{h}^{\#}\right)_i\left(\mathbf{I}_K - \Delta\mathbf{H}_t\right)\mathbf{H}^{\#}\mathbf{v}_{2,j}$$

where $\mathbf{h}_i^{\#}$ denotes the ith row of $\mathbf{H}^{\#}$, $\mathbf{w}_j$ and $\mathbf{v}_{2,j}$ denote jth columns of $\mathbf{W}$ and $\mathbf{V}_2$, respectively. Conditioned on $\mathbf{H}$, $\mathbf{V}_1$ and $\mathbf{W}$, $\left(\Delta\mathbf{W}_t\right)_{i,j}$ is a Gaussian random variable with mean equal to $-\mathbf{h}_i^{\#}\Delta\mathbf{H}_t\mathbf{w}_j$ and variance

$$\begin{aligned}
\sigma_{w,N}^2 &= \sigma_v^2\mathbf{h}_i^{\#}\left(\mathbf{I}_K - \Delta\mathbf{H}_t\right)\mathbf{H}^{\#}\left(\mathbf{H}^{\#}\right)^{\mathrm{H}}\left(\mathbf{I}_K - \Delta\mathbf{H}_t{}^{\mathrm{H}}\right)\left(\mathbf{h}_i^{\#}\right)^{\mathrm{H}} \\
&= \sigma_v^2\mathbf{h}_i^{\#}\left(\mathbf{I}_K - \Delta\mathbf{H}_t\right)\left(\mathbf{H}^{\mathrm{H}}\mathbf{H}\right)^{-1}\left(\mathbf{I}_K - \Delta\mathbf{H}_t{}^{\mathrm{H}}\right)\left(\mathbf{h}_i^{\#}\right)^{\mathrm{H}}.
\end{aligned}$$

The characteristic function of a complex Gaussian random variable is given by the following theorem.

Theorem 2: Let X_n be a complex Gaussian random variable with mean $m_{X,n}$ and variance $\sigma_{X,n}^2$, such that $\mathbb{E}(X_n - m_{X,n})^2 = 0$. Then, X_n can be seen as a two-dimensional random variable corresponding to its real and imaginary parts. The characteristic function of X_n is therefore given by:

$$\mathbb{E}\left[\exp\left(\jmath\Re(s^*X_n)\right)\right] = \exp\left(\jmath\Re\left(s^*m_{X,n}\right)\right)\exp\left(-\frac{1}{4}s^2\sigma_{X,n}^2\right).$$

Applying Theorem2, the conditional characteristic function of $\left(\Delta\mathbf{W}_t\right)_{i,j}$ can be written as:

$$\mathbb{E}\left[\exp\left(\jmath\Re\left(s^*\left(\Delta\mathbf{W}_t\right)_{i,j}\right)\right)|\mathbf{V}_1,\mathbf{H},\mathbf{W}\right] = \exp\left(-\jmath\Re\left(s^*\mathbf{h}_i^{\#}\Delta\mathbf{H}_t\mathbf{w}_j\right)\right)\exp\left(-\frac{1}{4}s^2\sigma_{w,N}^2\right).$$

To remove the condition expectation on $\mathbf{V}_1$ and $\mathbf{W}$, one should prove that $\sigma_{w,K}^2$ converges almost surely to a deterministic quantity. Actually, $\sigma_{w,N}^2$ can be expanded as follows:

$$\sigma_{w,N}^2 = \sigma^2\mathbf{h}_i^{\#}\left(\mathbf{h}_i^{\#}\right)^{\mathrm{H}} + \sigma^2\mathbf{h}_i^{\#}\Delta\mathbf{H}_t\left(\mathbf{H}^{\mathrm{H}}\mathbf{H}\right)^{-1}\left(\Delta\mathbf{H}_t\right)^{\mathrm{H}}\left(\mathbf{h}_i^{\#}\right)^{\mathrm{H}} - 2\sigma^2\Re\left(\mathbf{h}_i^{\#}\Delta\mathbf{H}_t\left(\mathbf{H}^{\mathrm{H}}\mathbf{H}\right)^{-1}\left(\mathbf{h}_i^{\#}\right)^{\mathrm{H}}\right).$$

Let

$$\begin{aligned}
A_{\sigma,N} &= \mathbf{h}_i^{\#}\Delta\mathbf{H}_t\left(\mathbf{H}^{\mathrm{H}}\mathbf{H}\right)^{-1}\left(\Delta\mathbf{H}_t\right)^{\mathrm{H}}\left(\mathbf{h}_i^{\#}\right)^{\mathrm{H}}, \\
\epsilon_{\sigma,N} &= \mathbf{h}_i^{\#}\Delta\mathbf{H}_t\left(\mathbf{H}^{\mathrm{H}}\mathbf{H}\right)^{-1}\left(\mathbf{h}_i^{\#}\right)^{\mathrm{H}}.
\end{aligned}$$

We study in the following the asymptotic behaviour of $A_{\sigma,N}$ and $\epsilon_{\sigma,N}$. But first, let us recall the following known results.

Theorem 3: Let $\mathbf{x} = [x_1,\cdots,x_N]^{\mathrm{T}}$ be a $N\times 1$ vector where the entries x_i are centered iid complex random variables with unit variance and finite fourth order. Let $\mathbf{A}$ be a determinsitic $N\times N$ complex matrix with bounded spectral norm. Then,

$$\frac{1}{N}\mathbf{x}^{\mathrm{H}}\mathbf{A}\mathbf{x} - \frac{1}{N}\mathrm{Tr}(\mathbf{A}) \longrightarrow 0 \qquad \text{almost surely.}$$

Theorem 4: Almost sure convergence of weighted averages

Let $\mathbf{a}_N = [a_1, \cdots, a_N]^{\mathrm{T}}$ be a sequence of $N \times 1$ determinstic of complex vectors with $\sup_N \frac{1}{N}\mathbf{a}_N^{\mathrm{H}}\mathbf{a}_N < \infty$. Let $\mathbf{x}_N$ be a $N \times 1$ random vector with iid entries such that $\mathbb{E}x_1 = 0$ and $\mathbb{E}|x_1| < \infty$. Therefore, $\frac{1}{N}\mathbf{a}_N^{\mathrm{H}}\mathbf{x}_N$ converges almost surely to zero as N tends to infinity.

Using Theorem3, it can be proved that:

$$A_{\sigma,N} - \frac{\sigma_v^2\left[(\mathbf{H}^{\mathrm{H}}\mathbf{H})^{-1}\right]_{i,i}}{N_1\sigma_P^2}\mathrm{Tr}\left(\mathbf{H}^{\mathrm{H}}\mathbf{H}\right)^{-1} \longrightarrow 0 \qquad \text{almost surely.}$$

Since $\frac{1}{K}\mathrm{Tr}\left(\mathbf{H}^{\mathrm{H}}\mathbf{H}\right)^{-1}$ converges asymptotically to $\frac{1}{c_2-1}$ as the dimensions go to infinity, we get:

$$A_{\sigma,N} - \frac{c_1(1+r)\sigma_v^2}{(c_2-1)\sigma_P^2}\left[(\mathbf{H}^{\mathrm{H}}\mathbf{H})^{-1}\right]_{i,i} \longrightarrow 0.$$

Note that Theorem3 can be applied since the smallest eigenvalue of the Wishart matrix $(\mathbf{H}^{\mathrm{H}}\mathbf{H})$ are almost surely uniformely bounded away from zero by $(1-\sqrt{c_2})^2 > 0$, [2]. Also, using theorem 4, we can prove that:

$$\epsilon_{\sigma,N} \longrightarrow 0 \qquad \text{almost surely.}$$

This leads to

$$\sigma_{w,N}^2 - \tilde{\sigma}_{w,N}^2 \longrightarrow 0 \qquad \text{almost surely.}$$

where $\tilde{\sigma}_{w,N}^2$ is given by:

$$\tilde{\sigma}_{w,N}^2 = \sigma_v^2\left[(\mathbf{H}^{\mathrm{H}}\mathbf{H})^{-1}\right]_{i,i} + \frac{c_1(1+r)\sigma_v^4}{(c_2-1)\sigma_P^2}\left[(\mathbf{H}^{\mathrm{H}}\mathbf{H})^{-1}\right]_{i,i}.$$

Conditioning on $\mathbf{H}$ and $\mathbf{W}$, the characteristic function satisfies asymptotically:

$$\mathbb{E}\left[\exp\left(\jmath\Re\left(s^*\left(\Delta\mathbf{W}_t\right)_{i,j}\right)\right)|\mathbf{H},\mathbf{W}\right] - \mathbb{E}\left[\exp\left(-\jmath\Re\left(s^*\mathbf{h}_i^{\#}\Delta\mathbf{H}_t\mathbf{w}_j\right)\right)|\mathbf{W},\mathbf{H}\right]\exp\left(-\frac{1}{4}s^2\tilde{\sigma}_{w,N}^2\right) \longrightarrow 0 \ \text{ almost surely.}$$

Also conditioning on $\mathbf{W}$ and $\mathbf{H}$, $\mathbf{h}_i^{\#}\Delta\mathbf{H}_t\mathbf{w}_j$ is a Gaussian random variable with zero mean and variance

$$\sigma_{m,N}^2 = \frac{\sigma_v^2}{N_1\sigma_P^2}\mathbf{h}_i^{\#}{\mathbf{w}_j}^{\mathrm{H}}\left(\mathbf{P}_t\mathbf{P}_t^{\mathrm{H}}\right)^{-1}\mathbf{w}_j\left(\mathbf{h}_i\right)^{\#}.$$

Since $\frac{1}{K}{\mathbf{w}_j}^{\mathrm{H}}\mathbf{w}_j \longrightarrow \sigma_w^2$ almost surely, we get that $\sigma_{m,N}^2$ converges almost surely to $\tilde{\sigma}_{m,N}^2$ where

$$\tilde{\sigma}_{m,N}^2 = \frac{c_1(1+r)\sigma_v^2\sigma_w^2}{\sigma_P^2}\left[(\mathbf{H}^{\mathrm{H}}\mathbf{H})^{-1}\right]_{i,i},$$

$$\mathbb{E}\left[\exp\left(-\jmath\Re\left(s^*\mathbf{h}_i^{\#}\Delta\mathbf{H}_t\mathbf{w}_j\right)\right)|\mathbf{W},\mathbf{H}\right] = \exp\left(-\frac{1}{4}s^2\sigma_{m,N}^2\right).$$

Finally, we obtain that conditionally on the channel:

$$\mathbb{E}\left[\exp\left(\jmath\Re\left(s^*\left(\Delta\mathbf{W}_t\right)_{i,j}\right)\right)\right] - \exp\left(-\frac{1}{4}s^2\left(\tilde{\sigma}_{m,N}^2 + \tilde{\sigma}_{w,N}^2\right)\right) \longrightarrow 0 \ \text{ almost surely.}$$

The proof is concluded by noticing that $\tilde{\sigma}_{m,N}^2 + \tilde{\sigma}_{w,N}^2 = \sigma_w^2\delta_t\left[(\mathbf{H}^{\mathrm{H}}\mathbf{H})^{-1}\right]_{i,i}$.

II. Data-dependent superimposed training scheme

We also stated in [1] the following theorem.

Theorem 5: Under the asymptotic regime, and conditioned on the channel, the post-processing noise experienced by the i-th antenna at each time j behaves asymptotically as a Gaussian mixture random variable, i.e:

$$\mathbb{E}\left[e^{\jmath\Re s^*\Delta\mathbf{W}_d(i,j)}\right] - \sum_{i=1}^{\mathcal{Q}} p_i e^{\left(\Re(\jmath s^*\alpha_i)\right)} e^{-\frac{|s|^2\delta_d\sigma_w^2\left[(\mathbf{H}^{\mathrm{H}}\mathbf{H})^{-1}\right]_{i,i}}{4}} \xrightarrow[N\to\infty]{} 0$$

where $\mathcal{Q}$ is the cardinal of the set of all possible values of $\overline{\mathbf{W}}(i,k) = c_1\sum_{k=1}^{1/c_1}\mathbf{W}(i,k)$ and p_i is the probability that $\overline{\mathbf{W}}(i,k)$ takes the value α_i. Moreover, δ_d is given by:

$$\delta_d = (1-c_1)\left(\frac{c_1\sigma_v^2}{\sigma_{P'}^2} + \frac{\sigma_v^2}{\sigma_{w'}^2} + \frac{c_1\sigma_v^4}{(c_2-1)\sigma_{P'}^2\sigma_{w'}^2}\right).$$

In the sequel, we provide the proof for this theorem. We recall in [1] that for the data-dependent scheme, the channel estimate is given by:

$$\widehat{\mathbf{H}}_d = \mathbf{H} + \Delta\mathbf{H}_d$$

where $\Delta\mathbf{H}_d = \mathbf{V}{\mathbf{P}_d}^{\mathrm{H}}\left(\mathbf{P}_d{\mathbf{P}_d}^{\mathrm{H}}\right)^{-1}$. After applying the zero forcing linear receiver, the effective post-processing noise $\Delta\mathbf{W}_d$ can be written as:

$$\begin{aligned}\Delta\mathbf{W}_d &= -\mathbf{W}\mathbf{J} - \mathbf{H}^{\#}\Delta\mathbf{H}_d\mathbf{W}\left(\mathbf{I}_N - \mathbf{J}\right) + \left(\mathbf{H}^{\#} - \mathbf{H}^{\#}\Delta\mathbf{H}_d\mathbf{H}^{\#}\right)\mathbf{V}\left(\mathbf{I}_N - \mathbf{J}\right)\\ &= -\mathbf{W}\mathbf{J} - \mathbf{H}^{\#}\Delta\mathbf{H}_d\mathbf{W}\left(\mathbf{I}_N - \mathbf{J}\right) + \mathbf{H}^{\#}\mathbf{V}\left(\mathbf{I}_N - \mathbf{J}\right) - \mathbf{H}^{\#}\Delta\mathbf{H}_d\mathbf{H}^{\#}\mathbf{V}\left(\mathbf{I}_N - \mathbf{J}\right).\end{aligned}$$

Hence

$$\left(\Delta\mathbf{W}_d\right)_{i,j} = -\tilde{\mathbf{w}}_i\mathbf{J}_j - \mathbf{h}_i^{\#}\mathbf{V}\mathbf{P}_d^{\mathrm{H}}\left(\mathbf{P}_d\mathbf{P}_d^{\mathrm{H}}\right)^{-1}\mathbf{W}\left(\mathbf{e}_j - \mathbf{J}_j\right) + \mathbf{h}_i^{\#}\mathbf{V}\left(\mathbf{e}_j - \mathbf{J}_j\right) - \mathbf{h}_i^{\#}\mathbf{V}{\mathbf{P}_d}^{\mathrm{H}}\left(\mathbf{P}_d\mathbf{P}_d^{\mathrm{H}}\right)^{-1}\mathbf{H}^{\#}\mathbf{V}\left(\mathbf{e}_j - \mathbf{J}_j\right)$$

where $\mathbf{e}_j$ and $\mathbf{J}_j$ denotes the jth columns of $\mathbf{I}_N$ and $\mathbf{J}$, respectively and $\tilde{\mathbf{w}}_i$ denotes the ith row of the matrix $\mathbf{W}$. Let $\mathbf{v}_1 = \mathbf{V}\left(\mathbf{e}_j - \mathbf{J}_j\right)$, and $\mathbf{v}_2 = \left[\mathbf{h}_i^{\#}\mathbf{V}\left(\mathbf{P}_d\mathbf{P}_d^{\mathrm{H}}\right)^{-1}\mathbf{p}_1^{\mathrm{H}}, \cdots, \mathbf{h}_i^{\#}\mathbf{V}\left(\mathbf{P}_d\mathbf{P}_d^{\mathrm{H}}\right)^{-1}\mathbf{p}_K^{\mathrm{H}}\right]^{\mathrm{T}}$. where $\mathbf{p}_1, \cdots, \mathbf{p}_K$ denote the K rows of $\mathbf{P}$. The vector $[\mathbf{v}_1^{\mathrm{T}}, \mathbf{v}_2^{\mathrm{T}}]^{\mathrm{T}}$ is a Gaussian vector. Since $\mathbb{E}\left[\mathbf{v}_1\mathbf{v}_2^{\mathrm{H}}\right] = 0$, $\mathbf{v}_1$ and $\mathbf{v}_2$ are independent. Moreover, $\mathbb{E}\left[\mathbf{v}_1\mathbf{v}_1^{\mathrm{H}}\right] = \sigma_v^2\left(1 - \frac{K}{N}\right)\mathbf{I}_N$, and $\mathbb{E}\left[(\mathbf{v}_2^{\mathrm{T}})^{\mathrm{H}}\mathbf{v}_2^{\mathrm{T}}\right] = \frac{\sigma_v^2}{N\sigma_{P'}^2}\left[\left(\mathbf{H}^{\mathrm{H}}\mathbf{H}\right)^{-1}\right]_{i,i}\mathbf{I}_K$.

Conditioning on $\mathbf{v}_2$, $\mathbf{H}$ and $\mathbf{W}$, $\left(\Delta\mathbf{W}_d\right)_{i,j}$ is a Gaussian random variable with mean equal to $-\tilde{\mathbf{w}}_i\mathbf{J}_j - \mathbf{v}_2^{\mathrm{T}}\mathbf{W}\left(\mathbf{e}_j - \mathbf{J}_j\right)$ and variance $\sigma_{w_d,N}^2$ equal to:

$$\begin{aligned}\sigma_{w_d,N}^2 &= \mathbb{E}\left[\left(\mathbf{h}_i^{\#} - \mathbf{v}_2^{\mathrm{T}}\mathbf{H}^{\#}\right)\mathbf{v}_1\mathbf{v}_1^{\mathrm{H}}\left(\left(\mathbf{h}_i^{\#}\right)^{\mathrm{H}} - \left(\mathbf{H}^{\#}\right)^{\mathrm{H}}(\mathbf{v}_2^{\mathrm{T}})^{\mathrm{H}}\right)|\mathbf{v}_2\right]\\ &= \mathbb{E}\left[\mathbf{h}_i^{\#}\mathbf{v}_1\mathbf{v}_1^{\mathrm{H}}\left(\mathbf{h}_i^{\#}\right)^{\mathrm{H}}\right] + \mathbb{E}\left[\mathbf{v}_2^{\mathrm{T}}\mathbf{H}^{\#}\mathbf{v}_1\mathbf{v}_1^{\mathrm{H}}\left(\mathbf{H}^{\#}\right)^{\mathrm{H}}(\mathbf{v}_2^{\mathrm{T}})^{\mathrm{H}}\right] - 2\mathbb{E}\left[\Re\left(\mathbf{v}_2^{\mathrm{T}}\mathbf{H}^{\#}\mathbf{v}_1\mathbf{v}_1^{\mathrm{H}}\left(\mathbf{h}_i^{\#}\right)^{\mathrm{H}}\right)\right]\\ &= (1 - \frac{K}{N})\sigma_v^2\left[\left(\mathbf{H}^{\mathrm{H}}\mathbf{H}\right)^{-1}\right]_{i,i} + \sigma_v^2(1 - \frac{K}{N})\mathbf{h}_i^{\#}\mathbf{v}_2^{\mathrm{T}}\left(\mathbf{H}^{\mathrm{H}}\mathbf{H}\right)^{-1}(\mathbf{v}_2^{\mathrm{T}})^{\mathrm{H}}\left(\mathbf{h}_i^{\#}\right)^{\mathrm{H}} - 2(1 - \frac{K}{N})\Re\left(\mathbf{v}_2^{\mathrm{T}}\mathbf{H}^{\#}\left(\mathbf{h}_i^{\#}\right)^{\mathrm{H}}\right).\end{aligned}$$

Using the same techniques as before, it can be proved that:

$$(1 - \frac{K}{N})\sigma_v^2\mathbf{h}_i^{\#}\mathbf{v}_2^{\mathrm{T}}\left(\mathbf{H}^{\mathrm{H}}\mathbf{H}\right)^{-1}(\mathbf{v}_2^{\mathrm{T}})^{\mathrm{H}}\left(\mathbf{h}_i^{\#}\right)^{\mathrm{H}} - \frac{c_1\left(1 - c_1\right)\sigma_v^4}{\left(c_2 - 1\right)\sigma_{P'}^2}\left[\left(\mathbf{H}^{\mathrm{H}}\mathbf{H}\right)^{-1}\right]_{i,i} \longrightarrow 0 \text{ almost surely}$$

and also that,

$$\mathbf{v}_2^{\mathrm{T}}\mathbf{H}^{\#}\left(\mathbf{h}_i^{\#}\right)^{\mathrm{H}} \longrightarrow 0 \text{ almost surely.}$$

Therefore,

$$\sigma_{w_d,N}^2 - \tilde{\sigma}_{w_d,N}^2 \longrightarrow 0 \text{ almost surely}$$

where,

$$\tilde{\sigma}_{w_d,N}^2 = \left(\sigma_v^2(1 - c_1) + \frac{c_1(1 - c_1)\sigma_v^4}{(c_2 - 1)\sigma_{P'}^2}\right)\left[\left(\mathbf{H}^{\mathrm{H}}\mathbf{H}\right)^{-1}\right]_{i,i}.$$

Consequently,

$$\mathbb{E}\left[\exp\left(\jmath\Re\left(s^*\left(\Delta\mathbf{W}\right)_{i,j}\right)\right)|\mathbf{H}, \mathbf{W}, \mathbf{v}_2\right] = \mathbb{E}\left[\exp\left(-\jmath\Re\left(s^*\tilde{\mathbf{w}}_i\mathbf{J}_j + s^*\mathbf{v}_2^{\mathrm{T}}\mathbf{W}\left(\mathbf{e}_j - \mathbf{J}_j\right)\right)\right)|\mathbf{W}, \mathbf{v}_2\right]\exp\left(-\frac{1}{4}s^2\tilde{\sigma}_{w_d,N}^2\right).$$

Conditioning on $\mathbf{W}$ and $\mathbf{H}$, $\tilde{\mathbf{w}}_i\mathbf{J}_j + \mathbf{v}_2^{\mathrm{T}}\mathbf{W}\left(\mathbf{e}_j - \mathbf{J}_j\right)$ is a Gaussian random variable with mean equal to $\tilde{\mathbf{w}}_i\mathbf{J}_j$ and variance $\sigma_{w_m,N}^2$ given by:

$$\begin{aligned}\sigma_{m_d,N}^2 &= \mathbb{E}\left[\mathbf{v}_2^{\mathrm{T}}\mathbf{W}\left(\mathbf{e}_j - \mathbf{J}_j\right)\left(\mathbf{e}_j^{\mathrm{H}} - \mathbf{J}_j^{\mathrm{H}}\right)\mathbf{W}^{\mathrm{H}}(\mathbf{v}_2^{\mathrm{H}})^{\mathrm{T}}|\mathbf{W}, \mathbf{H}\right]\\ &= \frac{\sigma_v^2}{N\sigma_{P'}^2}\left[\left(\mathbf{H}^{\mathrm{H}}\mathbf{H}\right)^{-1}\right]_{i,i}\left[\left(\mathbf{H}^{\mathrm{H}}\mathbf{H}\right)^{-1}\right]_{i,i}\left(\mathbf{e}_j^{\mathrm{H}} - \mathbf{J}_j^{\mathrm{H}}\right)\mathbf{W}\mathbf{W}^{\mathrm{H}}\left(\mathbf{e}_j - \mathbf{J}_j\right).\end{aligned}$$

Using Theorem 3, we can easily prove that:

$$\sigma_{m_d,N}^2 - \tilde{\sigma}_{m_d,N}^2 \longrightarrow 0 \text{ almost surely,}$$

where

$$\tilde{\sigma}_{m_d,N}^2 = \frac{(1 - c_1)\sigma_w^2\sigma_v^2}{\sigma_{P'}^2}\left[\left(\mathbf{H}^{\mathrm{H}}\mathbf{H}\right)^{-1}\right]_{i,i}.$$

Conditioning only on $\mathbf{H}$, the conditional characteristic function satisfies:

$$\mathbb{E}\left[\exp\left(\jmath\Re\left(s^*\left(\Delta\mathbf{W}_d\right)_{i,j}\right)\right)|\mathbf{H}\right] - \mathbb{E}\left[\exp\left(-\jmath\Re\left(s^*\tilde{\mathbf{w}}_i\mathbf{J}_j\right)\right)\right]\exp\left(-\frac{1}{4}s^2\left(\tilde{\sigma}_{w_d,N}^2 + \tilde{\sigma}_{m_d,N}^2\right)\right) \longrightarrow 0.$$

Giving the structure of the matrix $\mathbf{J}$, $\tilde{\mathbf{w}}_i\mathbf{J}_j$ involves the average of $\frac{1}{c_1}$ symmetric independent and identically distributed discrete random variables, and therefore,

$$\mathbb{E}\left[\exp\left(-j\Re\left(s^*\tilde{\mathbf{w}}_i\right)\right)\right]=\sum_{i=1}^{\mathcal{Q}}p_i\exp\left(\jmath\Re\left(s^*\alpha_i\right)\right)$$

where $\mathcal{Q}$ is the set of all possible values of $\overline{\mathbf{W}}_{i,k}=c_1\sum_{i=1}^{\frac{1}{c_1}}\mathbf{W}_{i,k}$ and p_i is the probability that $\overline{\mathbf{W}}_{i,k}$ takes the value α_i. Consequently;

$$\mathbb{E}\left[\exp\left(\jmath\Re\left(s^*\left(\Delta\mathbf{W}_d\right)_{i,j}\right)\right)|\mathbf{H}\right]=\sum_{i=1}^{\mathcal{Q}}p_i\exp\left(\jmath\Re\left(s^*\alpha_i\right)\right)\exp\left(-\frac{1}{4}s^2\left(\tilde{\sigma}_{m_d,N}^2+\sigma_{w_d,N}^2\right)\right).$$

We conclude the proof by noting that

$$\tilde{\sigma}_{m_d,N}^2+\sigma_{w_d,N}^2=\sigma_w^2\left[\left(\mathbf{H}^{\mathrm{H}}\mathbf{H}\right)^{-1}\right]_{i,i}\delta_d.$$

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