Quasi-Convexity of the Asymptotic Channel MSE in Regularized Semi Blind Estimation

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Abstract—In this paper, the quasi-convexity of a sum of quadratic fractions in the form $\sum_{i=1}^{n} \frac{1+c_i x^2}{(1+d_i x)^2}$ is demonstrated where c_i and d_i are strictly positive scalars, when defined on the positive real axis $\mathbb{R}^+$. It will be shown that this quasi-convexity guarantees it has a unique local (and hence global) minimum. Indeed, this problem arises when considering the optimization of the weighting coefficient in regularized semi-blind channel identification problem, and more generally, is of interest in other contexts where we combine two different estimation criteria. Note that V. Buchoux *et al.* have noticed by simulations that the considered function has no local minima except its unique global minimum but this is the first time this result, as well as the quasi-convexity of the function is proved theoretically.

Index Terms—Asymptotic analysis, channel estimation, exponential polynomial, minimum MSE, quasi-convexity, regularization, semi-blind estimation.

I. INTRODUCTION

MANY parameter estimation techniques use combined criteria to exploit different features or properties of the considered signals and hence improve the estimation performance. Examples of such combined techniques include the blind source separation (BSS) method in [1], and the blind equalization method in [2] where second and higher order statistics-based criteria are combined to restore the source signals, and the channel equalization and offset estimation technique in [3], where again two criteria based on two different features of the transmit signals are jointly used to improve the receiver performance.

In [4], a similar approach is used for channel shortening in OFDM systems, and in [5] and [6], semi-blind channel identification methods are considered where data-aided and blind techniques are combined together to shorten the training sequence while preserving a high channel estimation quality.

When combining two criteria, one uses a weighting parameter that needs to be optimized. In [7], the weighting coefficient is optimized in such a way the asymptotic mean square error (MSE) of the channel estimate is minimum.

The latter is shown to be a nonlinear function and its optimization in [7] is done numerically using a line search algorithm.

In this paper, we demonstrate that the previous asymptotic MSE function is quasi-convex which provides a guarantee that numerical optimization always leads to the desired optimal weighting parameter value.

Moreover, one can observe that many asymptotic MSE functions have similar forms to that in [7] and thus, we believe the result given in this paper might be extended and adapted to other problems, where an optimal weighting coefficient is needed to combine two contrast functions. As an example, we can cite the case where a contrast function is linearly combined with an MSE criterion. Referring to the work in [8], we can easily show that in this case, the expression of the asymptotic MSE has the same form as the one described in [7].

From the mathematics point of view, our work can be viewed as a contribution to the study of the roots of real exponential polynomials. It should be noted that this issue has been studied for general cases in [9], where interesting results about the number of roots of real exponential polynomials with real frequencies have been presented. Unfortunately, these results yield a loose bound for the number of roots of the considered exponential polynomial and thus are of no interest for our particular case given by (12). We have provided in our work an original proof that takes into consideration the specifications of the considered exponential polynomial.

The paper is organized as follows. Section II summarizes the results in [7] and shows that the considered MSE optimization problem can be cast into the optimization problem of a sum of quadratic fractions of the form

$$\sum_{i=1}^{n} \frac{1+c_i x^2}{(1+d_i x)^2}, c_i > 0, d_i > 0.$$

Sections III and IV are completely devoted to the derivation and the proof of the quasi-convexity of the asymptotic MSE function. In particular, Section III contains some basic notions and results about quasi-convex functions. Conclusion and final remarks are given in Section V.

Notation: Operators ^{H}, $^{-1}$ and Tr denote Hermitian, matrix inversion and trace operators. Moreover, the real and imaginary parts of a complex z are denoted respectively by $\text{Re}(z)$ and $\text{Im}(z)$.

II. REGULARIZED SEMI-BLIND CHANNEL ESTIMATION

In many signal processing applications, the major problem is to find out how to estimate some parameters at a low cost and with a good accuracy. The best estimate we can have is obtained by taking into account all the information that we can get about the desired parameter. This approach involves in general high
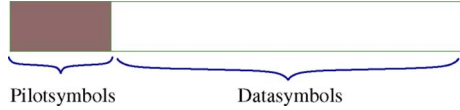
Fig. 1.   Time-multiplexed training scheme.

computational complexity, thus restricting its interest to only theoretical issues.

Actually, in practice, suboptimal approaches retain only one kind of information on which they are based to derive a minimization problem with only a single criterion. An intermediate approach that is based on linearly combining competitive criteria has been recently proposed in many signal processing applications. For instance, in [4], the channel shortening is being improved by linearly combining the null tones criterion with that of the guard interval. Also in the context of estimating sparse parameter vectors, $\mathcal{L}_p$ (where $p < 1$) quasi-norms are often linearly combined to standard statistical criteria, thus allowing to take into account the sparsity of the desired solution [10].

The optimal selection of the regularizing parameter that makes the best trade-off between two different criteria is important for the considered parameter estimation problem. To the best of our knowledge, the minimization of the mean square estimation error with respect to the regularizing coefficient has been only analysed rigorously in [7], in the context of semi-blind channel estimation.

Since we will heavily rely on the asymptotic expression derived in [7], it may be illuminating to provide a brief overview on the regularized semi-blind estimation technique.

Regularized semi-blind estimation technique combines blind and training-based criteria. They have been introduced first in the context of Single Input Multiple Output (SIMO) systems. In this case, if s_k denotes the unit power transmitted signal, the vector received by the N receiving antennas $\mathbf{y}_k$ is given by:

$$\mathbf{y}_k = \sum_{l=0}^{L} \mathbf{h}_l s_{k-l} + \mathbf{v}_k \tag{1}$$

where $\mathbf{h}_l$ is the $N \times 1$ vector of the l-th tap of the channel impulse response and $\mathbf{v}_k$ denotes the additive Gaussian noise. We assume that each frame is composed of training and data period, (see Fig. 1). The training period corresponds to the transmission of ℓ known symbols which are often referred to as pilots, whereas the data period corresponds to the transmission of p data symbols.

The blind criterion is based on the statistical properties of the received signal in the data period and can be put on the form

$$\min_{\|\mathbf{h}\|=1} \mathbf{h}^{\mathrm{H}} \mathbf{Q} \mathbf{h} \tag{2}$$

where $\mathbf{h}$ is the channel parameter and $\mathbf{Q}$ is a matrix that depends solely on the statistical properties of the received signal. On the other hand, the training-based criterion can be expressed as

$$\min_{\mathbf{h}} \|\mathbf{y} - \mathbf{S}\mathbf{h}\|^2 \tag{3}$$

where $\mathbf{y}$ is the received signal and $\mathbf{S}$ is a matrix that depends on the pilot symbols. In contrast to blind estimation methods, training-based techniques are more sensitive to noise and en-

tail inefficient bandwidth utilization. However, blind methods are more complex, estimate the channel only up to a scalar ambiguity and are often nonrobust to modelization errors (e.g., channel order overestimation errors) [11]). For these reasons, it might be interesting to combine linearly both criteria so as to resolve the drawbacks inherent to blind and training-based techniques. Hence, the semi-blind estimate is the one that minimizes

$$\min_{\mathbf{h}} \|\mathbf{y} - \mathbf{S}\mathbf{h}\|^2 + \lambda p \mathbf{h}^{\mathrm{H}} \mathbf{Q} \mathbf{h} \tag{4}$$

where $\lambda > 0$ is the regularizing coefficient and p is the length of the information sequence. Note that the semi-blind approach in (4) outperforms the blind approach in (2) and the nonblind approach in (3) only if the regularizing scalar λ is chosen properly. In particular, this would be the case if λ is selected in such a way the asymptotic estimation error variance is minimized. Our result is also useful to derive a relation between the optimal MSE* and the percentage of training symbols which can be adjusted to achieve a target MSE performance.

It has been proved in [7] that the trace of the asymptotic[1] estimation mean-square error (MSE) is proportional to

$$\mathrm{MSE} \propto \mathrm{Tr}\left\{ (\mathbf{I} + \lambda\gamma\mathbf{Q})^{-1} \left(\mathbf{I} + \lambda^2\gamma\mathcal{M}(\mathbf{h}) \right) (\mathbf{I} + \lambda\gamma\mathbf{Q})^{-1} \right\}$$

where $\gamma = \frac{p}{\ell}$, and $\mathcal{M}(\mathbf{h})$ is a Hermitian matrix that has the same row and column space as $\mathbf{Q}$ (meaning that if $\mathbf{Q} = \mathbf{U}\mathbf{D}\mathbf{U}^{\mathrm{H}}$ is the eigenvalue decomposition of $\mathbf{Q}$, $\mathcal{M}(\mathbf{h})$ writes as $\mathcal{M}(\mathbf{h}) = \mathbf{U}\mathbf{A}\mathbf{U}^{\mathrm{H}}$, where $\mathbf{A}$ is a given Hermitian matrix.).

Using the eigenvalue decomposition of $\mathbf{Q}$, it can be easily verified that the MSE is proportional to

$$\mathrm{MSE} \propto \sum_i \frac{1 + \lambda^2 \gamma a_{ii}}{(1 + \lambda\gamma d_{ii})^2} \tag{5}$$

where $a_{ii} > 0$ (resp. $d_{ii} > 0$) denote the diagonal elements of $\mathbf{A}$ (resp. the non zero diagonal elements of $\mathbf{D}$).

Note that in [7] and [12], it was noticed by simulations that the MSE has a unique local (global) minimum with respect to λ, but to the best to our knowledge, until now, this result has not been proved in any previous work.

III. QUASI-CONVEXITY OF THE MEAN SQUARE ERROR

For the reader convenience, we recall hereafter the definition and also some results about quasi-convex functions (we refer the reader to [13] for further information).

Definition 1: A real valued function f is said to be quasi-convex if its domain of definition and all its sublevel sets:

$$S_\alpha = \{x \in \mathbf{dom}f \mid f(x) \leq \alpha\}$$

for $\alpha \in \mathbb{R}$, are convex, where $\mathbf{dom}f$ denotes the set over which the function f is defined.

Examples of quasi-convex functions To illustrate this concept, we provide in Fig. 2 some examples of quasi-convex functions. As we can see, we note that a concave and also a non convex function can be also quasi-convex.

Like convex functions, quasi-convex functions satisfy a modified Jensen inequality which is given by the following theorem.

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[1]Asymptotic refers to the case where $p \to \infty$, $\ell \to \infty$ and the ratio $\frac{p}{\ell} \to \gamma$.
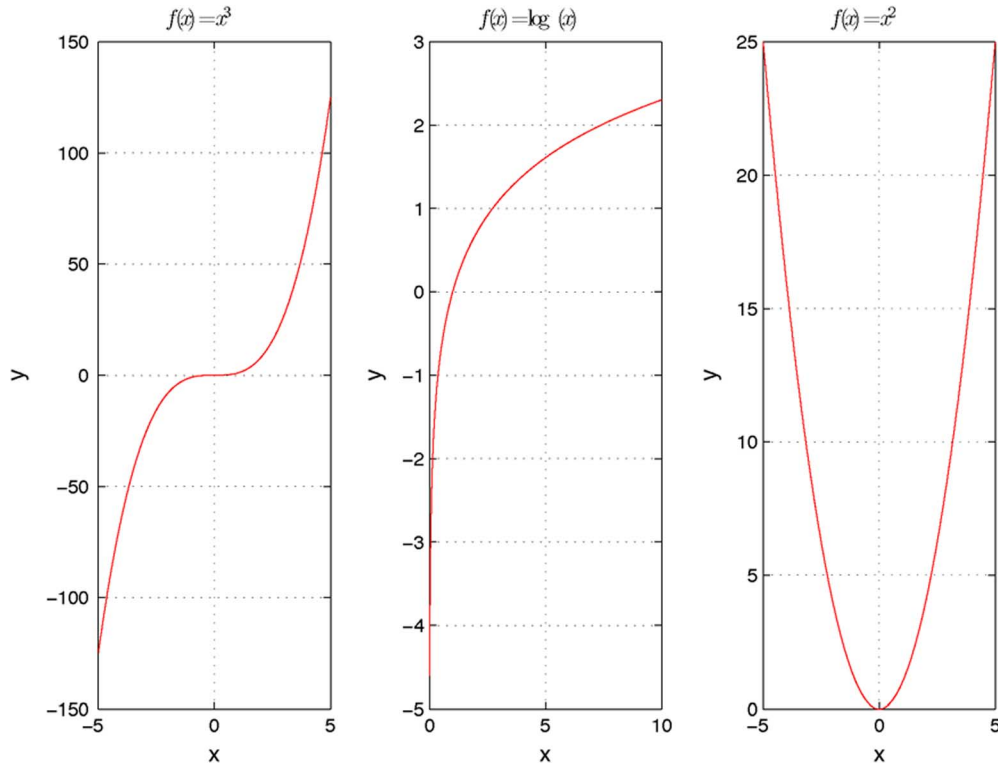
Fig. 2.   Examples of quasi-convex functions.

Theorem 1: A function f is quasi-convex if and only if $\mathbf{dom}f$ is convex and for all $x, y \in \mathbf{dom}(f)$ and $0 \le \theta \le 1$

$$f(\theta x + (1 - \theta)y) \le \max\{f(x), f(y)\}.$$

Clearly, the quasi-convexity generalizes the notion of convexity in the sense that the class of quasi-convex functions is larger than and includes the class of convex functions. Also, in most cases, quasi-convex functions inherit the nice properties of convex functions including the absence of local minimum as stated in the following theorem.

Theorem 2: Let f be a quasi-convex function. Then every local minimum is a global minimum or f is constant in a neighborhood of this local minimum.

Consequently, if a quasi-convex function f is non constant over any given interval (which is the case for the sum of quadratic functions we consider), then each local minimum is also a global minimum. Moreover, this global minimum (whenever it exists) is unique for real valued functions. To prove the non existence of local minima besides the global one, we use often the following second-order condition.

Theorem 3: Let f be a real function which is twice derivable. If f satisfies:

$$\forall c \quad \text{such that} \quad f'(c) = 0, f''(c) > 0,$$

then, f is quasi-convex, and each local minimum is a global minimum.

Next we state our main result regarding the unimodality of the asymptotic MSE then we prove it in the section after.

Theorem 4: Let c_i, d_i be two sequences of $n \in \mathbb{N}^*$ strictly positive reals. Then the derivative of

$$F_n(x) = \sum_{i=1}^{n} \frac{1 + c_i x^2}{(1 + d_i x)^2} \tag{6}$$

has a unique positive zero x_0 with $F_n^{(2)}(x_0) > 0$. Consequently, $F_n(x)$ is a quasi-convex function when its domain of definition is restricted to $\mathbb{R}^+$ and hence has a unique local (global) minimum on the positive real axis. In the sequel, we will omit the index n for notational simplicity so that F_n will be referred to as F.

To prove this theorem, we proceed in the following steps.
- First, we show that the number of positive real values of $F^{(k+1)}$ is larger or equal than that of $F^{(k)}$, where $F^{(k)}$ denotes the k-th derivative of F.
- We introduce the function G_k which has the same number of zeros as $F^{(k)}$ and prove that it converges uniformly to G_∞, over a compact set that contains all the zeros of $F^{(k)}$.
- Then we prove that G_∞ has a unique positive zero in that compact set.
- By applying Hurwitz theorem [14], we conclude that for large values of k, G_k is zero only once and that will be also the case of $F^{(k)}$.
- Finally, we prove that the second derivative of F is strictly positive when evaluated at the zero argument of F. Fig. 3 illustrates the shape of function F and its first and second order derivatives, for $n = 3$.

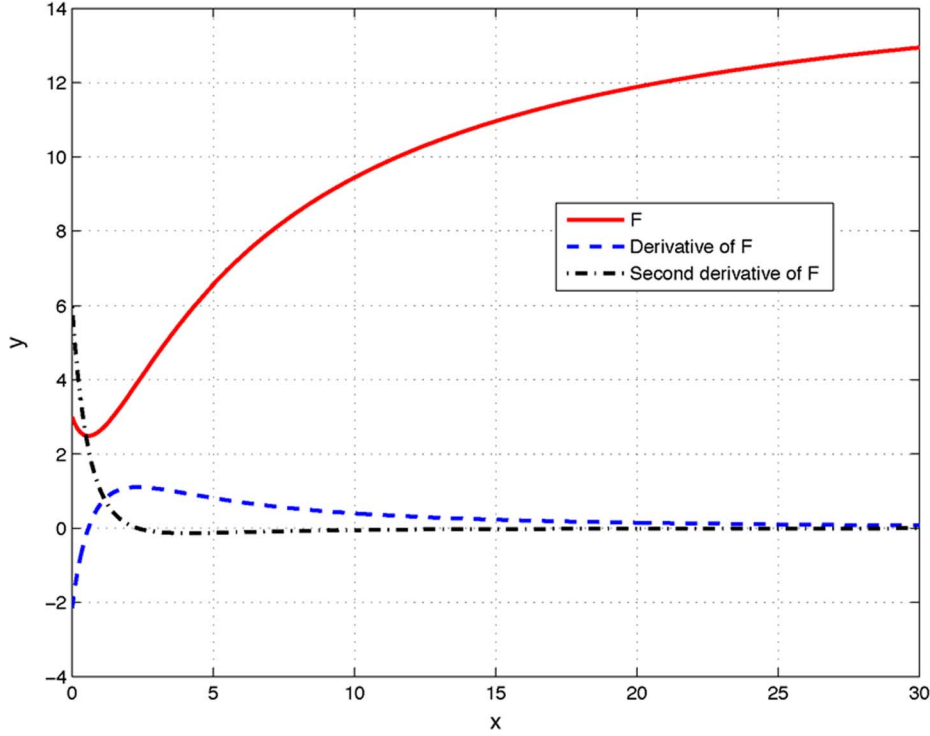The next section provides the details of all these steps and their proofs.

Fig. 3. Function F and its first and second-order derivatives.

IV. ANALYSIS AND PROPERTIES OF F

A. Closed-Form Expressions for the Derivatives of F

In this subsection, we provide a closed form expression for the k-th derivative of function F. We also show that the number of zeros of the k-th derivative is increasing with k.

Lemma 1: The k-th derivative of $F(x)$ $(k > 0)$ can be put on the following expression:

$$F^{(k)}(x) = (-1)^{k+1} \sum_{i=1}^{n} \frac{b_{i,k}x - a_{i,k}}{(1 + d_i x)^{k+2}} \quad (7)$$

where $a_{i,k}$ and $b_{i,k}$ are sequences of positive reals given by

$$\begin{cases} b_{i,k} = 2k! c_i d_i^{k-1} \\ a_{i,k} = 2k! c_i d_i^{k-1} \left(\frac{k+1}{2} \frac{d_i}{c_i} + \frac{k-1}{2d_i} \right). \end{cases}$$

Proof: See Appendix A. ∎

Given the previous expressions of $F^{(k)}$, we are able to prove our first step result concerning the increasing number of zeros of $F^{(k)}$. We have the following lemma.

Lemma 2: Let Z_k denote the number of zeros of the k-th derivative of F given by (7). Then $Z_{k+1} \geq Z_k$.

Proof: Let $x_1, \cdots, x_{Z_k}$ denote the zeros of the k-th derivative $F^{(k)}$ in $[0, \infty[$. Therefore, using Rolle's Theorem [15], $F^{(k+1)}$ has at least $Z_k - 1$ zeros $y_1, \cdots, y_{Z_k-1}$ where $x_i \leq y_i \leq x_{i+1}, i \in \{1, \cdots, Z_k - 1\}$. Since $\lim_{x \to +\infty} F^{(k)}(x) = 0$, there exist at least one zero of $F^{(k+1)}$ in $[x_{Z_k}, \infty[$. Consequently, the number of zeros of $F^{(k+1)}$ is at least equal to $Z^{(k)}$, i.e., $Z_{k+1} \geq Z_k$. ∎

B. Uniform Equivalence of G_k

In this subsection, we introduce an alternative function G_k that has the same number of positive valued zeros as $F^{(k)}$ and we provide its asymptotic equivalent expression. For that, let us start by providing a useful approximation of coefficient $a_{i,k}$ that will be used later to build the function G_k.

The Stirling formulae [16] provides us an equivalent[2] for $k!$

$$k! \sim \sqrt{2\pi k} \left(\frac{k}{e} \right)^k.$$

We can easily show that

$$\begin{aligned} a_{i,k} &\sim \sqrt{2\pi} k^{k+\frac{3}{2}} e^{-k} c_i d_i^{k-1} \left(\frac{d_i}{c_i} + \frac{1}{d_i} \right) \\ &\sim \sqrt{2\pi} k^{k+\frac{3}{2}} e^{-k} c_i d_i^{k+2} \left(\frac{1}{c_i d_i^2} + \frac{1}{d_i^4} \right). \end{aligned} \quad (8)$$

We recall that the overall quasi-convexity proof is based on studying the zeros of the function $F^{(k)}$ as k goes to infinity. Actually, one can show[3] that these zeros belong to the interval $[V_{\min}^k, V_{\max}^k]$, where $V_{\min}^k = \min_{i \in \{1, \cdots n\}} \frac{a_{i,k}}{b_{i,k}}$ and $V_{\max}^k = \max_{i \in \{1, \cdots n\}} \frac{a_{i,k}}{b_{i,k}}$. A lower bound for $V_{\min}^k$ and an upper bound for $V_{\max}^k$ can be easily computed and are given by

$$V_{\min}^k \geq k\tau_{\min} \quad (9)$$
$$V_{\max}^k \leq k\tau_{\max} \quad (10)$$

---

[2]Equivalence here means that $\frac{k!}{\sqrt{2\pi k}(\frac{k}{e})^k} \xrightarrow[[k \to \infty]]{} 1$

[3]Outside this interval, all the terms in the sum given by (7) have the same sign and hence $F^{(k)}$ cannot be zero.

where $\tau_{\min} = \min\limits_{i \in \{1,\cdots n\}} \dfrac{d_i}{2c_i}$ and $\tau_{\max} = \max\limits_{i \in \{1,\cdots n\}} \dfrac{d_i}{c_i} + \dfrac{1}{2d_i}$. The difficulty that we face is that the zeros of $F^{(k)}$ are of order k, thus making the analysis of the asymptotic behavior of function $F^{(k)}$ somehow delicate. To deal with this difficulty, another function, which we denote by G_k, and which brings back those zeros to a given fixed interval is introduced. This function will be studied over the interval of interest $[\tau_{\min}, \tau_{\max}]$.

Function G_k is defined as

$$G_k(x) = (-1)^{k+1} \sqrt{\frac{k}{2\pi}} e^k x^{k+2} F^{(k)}(kx). \qquad (11)$$

One can easily note that over $[\tau_{\min}, \tau_{\max}]$, $G_k(x)$ has the same number of zeros as $F^{(k)}$.

Clearly, the scaling of the variable x by factor k is introduced to bring back the roots of $F^{(k)}$ from the interval $[k\tau_{\min}, k\tau_{\max}]$ to the finite length interval $[\tau_{\min}, \tau_{\max}]$. The multiplicative function in (11) (i.e., $\sqrt{\frac{k}{2\pi}} e^k x^{k+2}$) is introduced to normalize the coefficient $a_{i,k}$ and $b_{i,k}$ and to approximate the denominator terms in (7) by exponential functions.

Substituting $F^{(k)}$ by its expression in (7), G_k writes as

$$G_k(x) = \sqrt{\frac{k}{2\pi}} e^k x^{k+2} \sum_{i=1}^{n} \frac{kb_{i,k}x - a_{i,k}}{(1 + kd_i x)^{k+2}}$$

$$= \sqrt{\frac{k}{2\pi}} e^k \sum_{i=1}^{n} \frac{a_{i,k} \left(\frac{kb_{i,k}}{a_{i,k}} x - 1 \right)}{k^{k+2} d_i^{k+2} \left(\frac{1}{kd_i x} + 1 \right)^{k+2}}$$

$$\triangleq \sum_{i=1}^{n} \frac{g_{i,k}(x)}{h_{i,k}(x)}$$

where $g_{i,k}(x) \triangleq \frac{\sqrt{k} e^k}{\sqrt{2\pi} d_i^{k+2} k^{k+2}} a_{i,k} \left(\frac{kb_{i,k}}{a_{i,k}} x - 1 \right)$ and $h_{i,k}(x) = \left(\frac{1}{kd_i x} + 1 \right)^{k+2}$.

In the following, we extend the domain of the function G_k to the rectangle R of $\mathbb{C}$ given by

$$\mathcal{R}_\epsilon = \{z = x + iy, x \in [\tau_{\min}, \tau_{\max}], -\epsilon \leq y \leq \epsilon\}$$

where ϵ is a constant real that will be specified later. Over this domain, the asymptotic equivalent of G_k is given by the following theorem: From the previously stated lemma, one can prove easily the following result:

Theorem 5: In the rectangle $\mathcal{R}_\epsilon$, G_k converges uniformly to G_∞ given by

$$G_\infty(z) = \sum_{i=1}^{n} c_i \left(\frac{1}{c_i d_i^2} + \frac{1}{d_i^4} \right) (V_{\infty,i} z - 1) e^{-\frac{1}{d_i z}}$$

where $V_{\infty,i} = \frac{2}{\frac{d_i}{c_i} + \frac{1}{d_i}}$.

Proof: See Appendix B. ∎

C. Zeros of the Uniform Limit of G_k

In this section, we prove that G_∞ has a unique positive real zero. This is a byproduct of the following theorem:

Theorem 6: Let a_i b_i and α_i three sequences of n strictly positive real scalars. Let f be the function given by:

$$f(x) = \sum_{i=1}^{n} (a_i x - b_i) e^{-\alpha_i x} \qquad (12)$$

Then f admits a unique real positive zero.

Proof: See Appendix C. ∎

By defining $f(z) = z G_\infty \left(\frac{1}{z} \right)$ and applying Theorem 6, we conclude that $G_\infty(z)$ has a unique real positive zero.

D. Application of Hurwitz Theorem

To prove that from a certain range of k G_k is zero only once at the real positive axis, we will rely on the following known result in complex analysis, [14]:

Theorem 7: Let $f_k(z)$ be a sequence of analytic functions in a compact $\mathbf{C}$. Assume that f_k converges uniformly to f in $\mathbf{C}$. Assume also that f has no zeros on the frontier $\partial \mathbf{C}$ of $\mathbf{C}$. Then, there exists $k_0 \in \mathbb{N}$ such that $\forall k \geq k_0$, f and f_k have the same number of zeros in $\mathbf{C}$.

Applying this theorem, we can deduce that, G_k will have a unique zero value in $\mathcal{R}_\epsilon$ as G_∞, where ϵ is chosen so that G_∞ has no zeros on the frontier of $\mathcal{R}_\epsilon$ and has no complex zeros besides its real positive zero. Since the number of zeros of G_k is increasing with respect to k, we conclude that all G_k and hence all $F^{(k)}$ have only a unique positive zero.

Let x_z be the unique positive zero argument of $F^{(1)}$. Since $F^{(1)}$ is negative in a neighborhood of zero, and $F^{(1)}$ has no zeros for $x \leq x_z$, $F^{(1)}$ is negative in the interval $[0, x_z]$. Therefore the function F is decreasing in $[0, x_z]$.

Since $F^{(1)}$ is positive for large value of x, $F^{(1)}$ must change its sign at x_z, and hence it is positive in the interval $[x_z, \infty[$. Consequently, F is increasing in $[x_z, \infty[$.

To sum up, we have established that in $[0, x_z]$, F is decreasing and in $[x_z, \infty[$ F is increasing. This guarantees that x_z is a minimum for F and hence $F^{(2)}(x_z) \geq 0$. In fact, $F^{(2)}(x_z)$ is strictly positive, since $F^{(1)}(x_z) = 0$ and $\lim\limits_{x \to \infty} F^{(1)}(x) = 0$ means that there exists $y_z \in]x_z, \infty[$ such that $F^{(2)}(y_z) = 0$ and hence $F^{(2)}(x_z) \neq 0$ (because $F^{(2)}$ has a unique zero).

V. CONCLUSION

In this paper, we have provided a rigorous proof for the quasi-convexity of the asymptotic MSE of the regularized semi-blind channel estimate.

More generally, we have proved that any function given by a finite sum of quadratic fractions $\frac{1 + cx^2}{(1 + dx)^2}, c, d > 0$ is a unimodal function over $\mathbb{R}^+$.

For our considered channel estimation problem, the previous result guarantees the absence of nondesired local minima of the MSE function when optimized with respect to the weighting coefficient.

Proof: Lemma 1 can be proved easily by induction on k. For $k = 1$, we have

$$
\begin{aligned}
F^{(1)}(x) &= \sum_{i=1}^{n} \frac{2c_i x(1+d_i x)^2 - 2d_i(1+c_i x^2)(1+d_i x)}{(1+d_i x)^4} \\
&= \sum_{i=1}^{n} \frac{(1+d_i x)(2c_i x(1+d_i x) - 2d_i(1+c_i x^2))}{(1+d_i x)^4} \\
&= \sum_{i=1}^{n} 2\frac{c_i x - d_i}{(1+d_i x)^3}.
\end{aligned}
$$

Let $k \in \mathbb{N}^*$. Assume that the result is true until order k. Hence, $F^{(k)}$ can be written as

$$
F^{(k)}(x) = (-1)^{k+1} \sum_{i=1}^{n} \frac{b_{i,k} x - a_{i,k}}{(1+d_i x)^{k+2}}.
$$

Therefore, we get the first equation shown at the bottom of the page, where $b_{i,k+1} = (k+1)b_{i,k}d_i$ and $a_{i,k+1} = (b_{i,k} + (k+2)d_i a_{i,k})$. Since $b_{i,k} = 2k!d_i^{k-1}c_i$, we get $b_{i,k+1} = 2(k+1)!d_i^k c_i$.

Also, we get the second equation shown at the bottom of the page. ∎

In this lemma, we propose to find the uniform-limit function for the function $G_k(z) = \sum_{i=1}^{n} \frac{h_{i,k}(z)}{g_{i,k}(z)}$ in the rectangle $\mathcal{R}_\epsilon$. For that, we will first begin by finding the uniform limit functions of $h_{i,k}$ and $g_{i,k}$.

Lemma 3: In the rectangle $\mathcal{R}_\epsilon$, the sequence of functions $(h_{i,k})_k$ converges uniformly to $h_{i,\infty}$ given by:

$$
h_{i,\infty}(z) = e^{\frac{1}{d_i z}}
$$

Also, the sequence of functions $(g_{i,k})_k$ converges uniformly to $g_{i,\infty}$ given by

$$
g_{i,\infty} = c_i \left(\frac{1}{c_i d_i^2} + \frac{1}{d_i^4} \right)(V_{\infty,i} z - 1)
$$

where $V_{\infty,i} = \lim_{k\to\infty} \frac{k b_{i,k}}{a_{i,k}} = \frac{2}{\frac{d_i}{c_i} + \frac{1}{d_i}}$.

Proof: The uniform convergence of $h_{i,k}$ to $h_{i,\infty}$ is a by-product of the following known result:

Lemma 4: Over a compact set the sequence function $(1+\frac{z}{n})^n$ converges uniformly to e^z.

The uniform convergence of $g_{i,k}$ to $g_{i,\infty}$ is obtained by using the asymptotic equivalent of $a_{i,k}$ given in (8). ∎

The uniform convergence of $h_{i,k}$ to $h_{i,\infty}$ and of $g_{i,k}$ to $g_{i,\infty}$ does not ensure the uniform convergence of $\frac{g_{i,k}}{h_{i,k}}$ to $\frac{g_{i,\infty}}{h_{i,\infty}}$.

Other extra conditions are needed as it will be noticed in the following lemma:

Lemma 5: Let f_k and g_k denote sequences of continuous functions over a compact C. Assume that g_k is bounded over

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$$
\begin{aligned}
F^{(k+1)}(x) &= (-1)^{k+1}\left(\sum_{i=1}^{n} \frac{b_{i,k}(1+d_i x)^{k+2}}{(1+d_i x)^{2k+4}} \right. \\
&\qquad\qquad \left. - \frac{(k+2)d_i(1+d_i x)^{k+1}(b_{i,k}x - a_{i,k})}{(1+d_i x)^{2k+4}} \right) \\
&= (-1)^{k+1} \sum_{i=1}^{n} \frac{b_{i,k} + (k+2)d_i a_{i,k} - (k+1)d_i b_{i,k} x}{(1+d_i x)^{k+3}} \\
&= (-1)^{k+2} \sum_{i=1}^{n} \frac{(k+1)b_{i,k}d_i x - (b_{i,k} + (k+2)d_i a_{i,k})}{(1+d_i x)^{k+3}} \\
&= (-1)^{k+2} \sum_{i=1}^{n} \frac{b_{i,k+1} x - a_{i,k+1}}{(1+d_i x)^{k+3}}
\end{aligned}
$$

$$
\begin{aligned}
a_{i,k+1} &= b_{i,k} + (k+2)d_i a_{i,k} \\
&= 2k!c_i d_i^{k-1} + 2k!(k+2)d_i^k c_i \left(\frac{k+1}{2}\frac{d_i}{c_i} + \frac{k-1}{2d_i} \right) \\
&= 2(k+1)!c_i d_i^k \left(\frac{1}{(k+1)d_i} + \frac{k+2}{k+1}\left(\frac{k+1}{2}\frac{d_i}{c_i} + \frac{k-1}{2d_i} \right) \right) \\
&= 2(k+1)!c_i d_i^k \left(\frac{k}{2d_i} + \frac{k+2}{2}\frac{d_i}{c_i} \right)
\end{aligned}
$$

C away from zero uniformly in k and in z, i.e., there exists a constant M such that

$$\forall k \in \mathbb{N}, \forall z \in C |g_k(z)| > M.$$

Assume also that f_k and g_k converge uniformly to f_∞ and g_∞. Then, $\frac{f_k}{g_k}$ converges uniformly to $\frac{f_\infty}{g_\infty}$ over the compact C.

Proof: Since f_k and g_k are continuous, their uniform limits f_∞ and g_∞ are also continuous. Therefore, there exists constant reals M_f, M_g such that

$$\forall z \in C, |f_\infty(z)| \le M_f \quad \text{and} \quad |g_\infty(z)| \le M_g.$$

Since for all $k \in \mathbb{N}$, $|g_k(z)| > M$, we have $|g_\infty(z)| > M$ To prove the uniform convergence of $\frac{f_k}{g_k}$ towards $\frac{f_\infty}{g_\infty}$, it is sufficient to prove that $\sup\limits_{z \in C} |\frac{f_k}{g_k} - \frac{f}{g}|$ converges to zero as k tends to infinity. We have f

$$\sup_{z \in C} \left| \frac{f_k}{g_k} - \frac{f}{g} \right| = \sup_{z \in C} \left| \frac{f_k g - f g_k}{g_k g} \right|$$
$$\le \sup_{z \in C} \left| \frac{f_k g - f g_k}{M^2} \right|$$
$$\le \frac{1}{M^2} \left(\sup_{z \in C} |f_k g - fg| + \sup_{z \in C} |fg - f g_k| \right)$$
$$\le \frac{1}{M^2} \left(M_g \sup_{z \in C} |f_k - f| + M_f \sup_{z \in C} |g_k - g| \right)$$
$$\xrightarrow[k \to \infty]{} 0$$

which proves that $\frac{f_k}{g_k}$ converges uniformly to $\frac{f_\infty}{g_\infty}$. ∎

Since $|h_{i,k}(z)| > 1$ over $\mathcal{R}_\epsilon$, $h_{i,k}$ satisfies the condition of Lemma 5. Applying this lemma on the functions $g_{i,k}$ and $h_{i,k}$, we prove that $\frac{g_{i,k}}{h_{i,k}}$ converges uniformly to $\frac{g_{i,\infty}}{h_{i,\infty}}$. Consequently, $G_k(z) = \sum\limits_{i=1}^{n} \frac{g_{i,k}(z)}{h_{i,k}(z)}$ converges uniformly over $\mathcal{R}_\epsilon$ to $G_\infty(z) = \sum\limits_{i=1}^{n} \frac{g_{i,\infty}(z)}{h_{i,\infty}(z)}$.

APPENDIX C
PROOF OF THEOREM 6

The proof is performed by induction on n. For $n = 1$, the result is straightforward. Let $n \in \mathbb{N}^*$ be a given integer, and assume that the result holds true for all $k \le n$, and all functions f of the form given by (12). Assume that there exists a_i, b_i and

α_i three sequences of $n + 1$ strictly positive real scalars such that the function

$$f(x) = \sum_{i=1}^{n+1} (a_i x - b_i) e^{-\alpha_i x}$$

admits more than one positive zero. Let x_1 be the first smallest zero of f on $\mathbb{R}^+$,

Without loss of generality, we can assume that all the α_i are two by two different and that $\alpha_{n+1} = \min\limits_{1 \le i \le n+1} \alpha_i$. Since f is strictly negative in zero and is positive for large values of x, f should change its sign at at least one zero. In the following we will consider only the case when f changes its sign at x_1. The other case coule be treated in the same way. Let x_2 be the second smallest zero of f on $\mathbb{R}^+$. Under this condition, we distinuish the following cases:

- f changes its sign at x_1 and at x_2.
- f changes its sign only at x_1.

For the both cases, we can prove that the second derivative of

$$g_m(x) = e^{\left(\alpha_{n+1} - \frac{1}{m}\right)x} f(x)$$
$$= \sum_{i=1}^{n} (a_i x - b_i) e^{-\left(\alpha_i - \alpha_{n+1} + \frac{1}{m}\right)x} + (a_{n+1} x - b_{n+1}) e^{-\frac{x}{m}},$$
for $m \in \mathbb{N}^*$

has three zeros. More particularly, we have the following:

Case 1: f changes its sign at x_1 and at x_2
Since $f(0) < 0$, $f(x) < 0$ for $x \le x_1$. Therefore, for $x \ge x_2$ and in the vicinity of x_2, $f(x) < 0$ for $x \ge x_2$. Since $f(x) > 0$ for x large enough, f should have a third zero $x_3 > x_2$.

For all integers m, we note that f and g_m have the same number of zeros. Using Rolles theorem, it can be proved that the derivative of g_m which we denote $g_m^{(1)}$ and which is given by (13)–(15), as shown at the bottom of the page, has at least three zeros, since $g_m^{(1)}(x)$ tends to zero as x tends to infinity.

Also again by using the Rolle's theorem, we conclude that the second derivative of g_m denoted by $g_m^{(2)}(x)$ has at least two zeros.

Case 2: f changes its sign at only one zero In this case, we can also prove that the first derivative of g_m has three zeros. Actually, at x_2, the first derivative of g_m must be also zero, since x_2 is a local minimum for f and hence for g_m. As g_m tends to zero when x tends to infinity, $g_m^{(1)}$ has two zeros between $]x_1, x_2[$ and $]x_2, \infty[$. Consequently, in total, $g_m^{(1)}$ has at least three zeros, and therefore, the second derivative of g_m denoted has at least two zeros.

$$g_m^{(1)}(x) = \sum_{i=1}^{n} \left[-\left(\alpha_i - \alpha_{n+1} + \frac{1}{m}\right) a_i x + b_i \left(\alpha_i - \alpha_{n+1} + \frac{1}{m}\right) \right. \tag{13}$$

$$\left. + a_i \right] \times e^{-\left(\alpha_i - \alpha_{n+1} + \frac{1}{m}\right)x} + a_{n+1} e^{-\frac{1}{m}x} - \frac{1}{m}(a_{n+1} x \tag{14}$$

$$- b_{n+1}) e^{-\frac{x}{m}} \tag{15}$$

Taking the derivative of (15), $g_m^{(2)}(x)$ writes as

$$
g_m^{(2)}(x) = \sum_{i=1}^n \left(\alpha_i - \alpha_{n+1} + \frac{1}{m} \right) \left[a_i \left(\alpha_i - \alpha_{n+1} + \frac{1}{m} \right) x \right.
$$
$$
\left. -2a_i - b_i \left(\alpha_i - \alpha_{n+1} + \frac{1}{m} \right) \right] e^{-\left(\alpha_i - \alpha_{n+1} + \frac{1}{m}\right)x}
$$
$$
- \frac{2}{m} a_{n+1} e^{-\frac{1}{m}x} + \frac{1}{m^2}(a_{n+1}x - b_{n+1})e^{-\frac{x}{m}}.
$$

Extending the definition domain of $g_m^{(2)}$ to $\mathbb{C}^+ = \{z = x + iy, x > 0\}$, we note that for every compact in $\mathbb{C}^+$, $g_m^{(2)}$ converges uniformly to g_∞ given by

$$
g_\infty(z) = \sum_{i=1}^n (\alpha_i - \alpha_{n+1})(a_i(\alpha_i - \alpha_{n+1})z - 2a_i
$$
$$
- b_i(\alpha_i - \alpha_{n+1}))e^{-(\alpha_i - \alpha_{n+1})z}.
$$

Let $\mathcal{C}$ be the contour corresponding to the rectangle

$$
R_\epsilon = \left\{ x + iy, x \in \left[\inf_{m,i} \frac{2a_i + b_i(\alpha_i - \alpha_{n+1}) + \frac{1}{m}}{a_i \left(\alpha_i - \alpha_{n+1} + \frac{1}{m} \right)}, \right. \right.
$$
$$
\left. \left. \sup_{m,i} \frac{2a_i + b_i(\alpha_i - \alpha_{n+1}) + \frac{1}{m}}{a_i \left(\alpha_i - \alpha_{n+1} + \frac{1}{m} \right)} \right], y \in [-\epsilon, \epsilon] \right\}.
$$

ϵ is chosen such that $|g_\infty|$ is bounded above zero in $\mathcal{C}$ and has no complex valued zeros. Then referring to Hurwitz theorem, g_∞ and $g_m^{(2)}$ will have the same number of zeros in R_ϵ for large enough values of m, which is in contradiction with the induction assumption.

REFERENCES

[1] M. Klajman and A. G. Constantinides, "A weighted mixed statistics algorithm for blind source separation," in *EUSIPCO*, 2002, vol. II, pp. 115–118.

[2] M. Corlay, P. Duhamel, and M. Charbit, "A combination of statistical and structural approaches in blind equalization," in *Proc. 34th Asilomar Conf. Signals, Systems and Computers*, 2000, vol. 2, pp. 1207–1211.

[3] A. Djebbar-Bouzidi, K. Abed-Meraim, and A. Djebbari, "Blind channel equalization and carrier frequency oset estimation for MC-CDMA systems using guard interval redundancy and excess codes," *Int. J. Electron. and Commun.*, vol. 63, pp. 220–225, 2009.

[4] T. Ben Jabeur, K. Abed-Meraim, and H. Boujemaa, "Blind channel shortening in OFDM system using nulltones and cyclic prefix," in *Proc. ICASSP*, Mar. 2008, pp. 3041–3044.

[5] G. Li and Z. Ding, "Semi-blind channel identification for individual data bursts in GSM wireless systems," *Signal Process.*, vol. 80, no. 10, pp. 2017–2031, Oct. 2000.

[6] A. A. Rontogiannis, K. Berberifis, A. Marava, and J. Palicot, "Efficient semi-blind estimation of multipath channel parameters via a delay decoupling optimization approach," *Signal Process.*, vol. 85, no. 12, pp. 2394–2411, 2005.

[7] V. Buchoux, O. Cappé, E. Moulines, and A. Gorokhov, "On the performance of semi-blind subspace-based channel estimation," *IEEE Trans. Signal Process.*, vol. 8, no. 6, pp. 1750–1759, Jun. 2000.

[8] J.-F. Cardoso and E. Moulines, "Minimum contrast estimation with application to array processing," in *Proc. Int. Conf. Acoustics Speech and Signal Processing*, 1993, vol. 4, pp. 384–387.

[9] F. Wielonsky, "A Rolle's theorem for real exponential polynomials in the complex domain," *J. des Mathématiq. Pures et Appliq.*, vol. 80, no. 4, pp. 389–408, May 2001.

[10] A. Aissa-El-Bey and K. Abed-Meraim, "Blind identification of sparse SIMO channels using maximum a posteriori approach," in *Proc. EU-SIPCO*, Lausanne, Switzerland, Aug. 2008.

[11] Y. Zeng and T. Sang Ng, "A blind MIMO channel estimation method robust to order overestimation," *Signal Process.*, vol. 84, no. 2, 2003.

[12] V. Buchoux, E. Moulines, and O. Cappé, "Estimation semi-autodidacte par méthode sous-espace pour systémes de réception multicapteurs," Actes du Colloque du GRETSI Sep. 1999.

[13] S. Boyd and L. Vandenberghe, *Convex Optimization*. New York: Cambridge University Press, 2004.

[14] R. Remmert, *Theory of Complex Functions, vol. 22 of Graduate Texts in Math.* Berlin, Germany: Springer-Verlag, 1991.

[15] C. R. J. Clapham, *Introduction to Mathematical Analysis*. New York: Routledge, 1973.

[16] J. Havil, *Gamma: Exploring Euler's Constant*. Princeton, NJ: Princeton University Press, 2003.

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