Estimating the Ergodic Capacity of Log-Normal Channels

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***Abstract*— In this paper, we first provide a very accurate estimation of the capacity of a single-input single-output system operating in a log-normal environment. Then, hinging on the fact that the sum of log-normal Random Variables (RV) is well approximated by another log-normal RV, we apply the obtained results to find the capacity of Maximum Ratio Combining and Equal Gain Combining in a log-normal environment. The capacity in an interference-limited environment is also investigated in this paper. The analytical expressions obtained match perfectly the capacity given by simulations.**

***Index Terms*— Information rates, log normal distributions.**

I. Introduction

Capacity of fading channels is one of the most investigated topics in the literature. The starting point of this investigation traces back to the beginning of the nineties with the seminal work of Lee [1], in which he derived the capacity of Rayleigh fading channels. Since then, a huge amount of research has addressed the capacity of different kind of fading channels (see [2], [3], and the references therein). However, despite all the research that was conducted, the capacity of the log-normal channel is left undiscovered. This dearth does not imply that the study of the log-normal capacity is less interesting. Indeed, wireless channels are usually modeled as log-normal (e.g. slow fading wireless channels). In the literature, to the best of the authors knowledge, no closed-forms were developed. Upper and lower bounds were developed in [7], but these bounds are loose for low SNR.

In this paper we will provide a very accurate approximation of the capacity of a Single-Input Single-Output (SISO) system operating in a log-normal environment. Then, relying on the fact that the sum of log-normal Random Variables (RV) is well approximated by another log-normal RV, we applied the obtained results to find the capacity of Maximum Ratio Combining (MRC) and Equal Gain Combining (EGC) in a log-normal environment. The capacity in an interference-limited environment will be also assessed in this paper.

This paper is organized as follows, in Section II we approximate the capacity of the log-normal channel in a SISO configuration. Section III extends these results to obtain the capacity with MRC and EGC. Section IV studies the capacity in an interference-limited environment. Section V gives some numerical examples and Section VI concludes the paper.

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II. The Capacity of a SISO Log-Normal Channel

The capacity (in Nats/Sec/Hz) of a log-normal channel is given by:

$$C=\frac{\xi}{\sigma\sqrt{2\pi}}\int_0^{+\infty}\frac{\ln(1+\gamma)}{\gamma}e^{-\frac{(\xi\ln\gamma-\mu)^2}{2\sigma^2}}d\gamma, \qquad (1)$$

where $\xi=\frac{10}{\ln(10)}$, σ is the logarithmic standard deviation of the fading process and $\mu=\Gamma_{\rm dB}-\frac{\sigma^2}{2\xi}$ where $\Gamma_{\rm dB}$ is the average SNR in dB. Next we will provide two accurate approximations for the capacity.

A. First approximation

After some manipulations in (1), C can be rewritten as

$$\begin{aligned} C &= I(\frac{\xi}{\sqrt{2}\sigma},\frac{\mu}{\sqrt{2}\sigma})+I(\frac{\xi}{\sqrt{2}\sigma},-\frac{\mu}{\sqrt{2}\sigma}) \\ &+ \frac{\xi}{\sigma\sqrt{2\pi}}\int_0^{+\infty}xe^{-\frac{(\xi x-\mu)^2}{2\sigma^2}}dx \qquad (2)\\ &= I(\frac{\xi}{\sqrt{2}\sigma},\frac{\mu}{\sqrt{2}\sigma})+I(\frac{\xi}{\sqrt{2}\sigma},-\frac{\mu}{\sqrt{2}\sigma}) \\ &+ \frac{\sigma}{\xi\sqrt{2\pi}}e^{\frac{-\mu^2}{2\sigma^2}}+\frac{\mu}{2\xi}\operatorname{erfc}(-\frac{\mu}{\sqrt{2}\sigma}), \qquad (3)\end{aligned}$$

where $I(a,b)$ is given by

$$I(a,b)=\frac{a}{\sqrt{\pi}}\int_0^1\frac{\ln(1+x)}{x}\exp(-(a\ln(x)-b)^2)dx. \qquad (4)$$

$I(a,b)$ can not be expressed in closed-form, however, it is possible to approximate it very accurately. Indeed, for $0\le x\le 1$, we have the following approximation [Eq. (4.1.44) in [6][1]]

$$\ln(1+x)=\sum_{k=1}^{8}a_kx^k+\varepsilon(x), \qquad (5)$$

where $|\varepsilon(x)|<3\times10^{-8}$ and a_k are constants defined in [6]. Consequently, an estimate for $I(a,b)$ is as follows

$$\begin{aligned}\hat{I}(a,b) &= \frac{a}{\sqrt{\pi}}\sum_{k=1}^{8}a_k\int_0^1x^{k-1}\exp(-(a\ln(x)-b)^2)dx \qquad (6)\\ &= \frac{e^{-b^2}}{2}\sum_{k=1}^{8}a_k\operatorname{erfcx}(\frac{k}{2a}+b), \qquad (7)\end{aligned}$$

where $\operatorname{erfcx}(x)=e^{x^2}\operatorname{erfc}(x)$ is the scaled complementary error function and is a built-in MATLAB function. The error that results from this approximation can be bounded as follows

$$|I(a,b)-\hat{I}(a,b)|<\frac{3\times10^{-8}}{2}\operatorname{erfc}(b). \qquad (8)$$

[1]Note that this formula does not correspond exactly to the Taylor series expansion of the $\ln(1+x)$, indeed $a_k\neq\frac{(-1)^{k+1}}{k}$. It should be noted also that a very similar expression (Eq. (4.1.43) in [6]) with only 5 terms is also available.

An approximation of the capacity will be therefore

$$\hat{C}=\frac{e^{-\frac{\mu^2}{2\sigma^2}}}{2}\sum_{k=1}^{8}a_k\left[\operatorname{erfcx}(\frac{\sigma k}{\xi\sqrt{2}}+\frac{\mu}{\sqrt{2}\sigma})+\operatorname{erfcx}(\frac{\sigma k}{\xi\sqrt{2}}-\frac{\mu}{\sqrt{2}\sigma})\right]$$
$$+\quad\frac{\sigma}{\xi\sqrt{2\pi}}e^{\frac{-\mu^2}{2\sigma^2}}+\frac{\mu}{2\xi}\operatorname{erfc}(-\frac{\mu}{\sqrt{2}\sigma}). \tag{9}$$

And the error on the capacity can be bounded as follows

$$|C-\hat{C}|<\frac{3}{2}10^{-8}(\operatorname{erfc}(\frac{\mu}{\sqrt{2}\sigma})+\operatorname{erfc}(-\frac{\mu}{\sqrt{2}\sigma}))=3\cdot 10^{-8}. \tag{10}$$

B. Second approximation

By definition, the capacity of the log-normal channel is

$$C=E[\ln(X)], \tag{11}$$

where X is equal to $1+\gamma$. In this second approximation, we approximate X by a log-normal RV, with a logarithmic mean equal to μ_X, so that $\ln(X)$ will be seen as a gaussian RV with a mean equal to $\frac{\mu_X}{\xi}$. Then, an approximation of the capacity is

$$C\approx\frac{\mu_X}{\xi}. \tag{12}$$

μ_X can be determined by a moment matching technique (the so called Fenton-Wilkinson method [4] as we will see in the next section). After some manipulations, the following approximation of the capacity is obtained

$$C\approx\ln\left(\frac{(1+e^{\frac{\mu}{\xi}+\frac{\sigma^2}{2\xi^2}})^2}{\sqrt{1+2e^{\frac{\mu}{\xi}+\frac{\sigma^2}{2\xi^2}}+e^{\frac{2\mu}{\xi}+\frac{2\sigma^2}{\xi^2}}}}\right), \tag{13}$$

where μ and σ are, respectively, the logarithmic mean and variance of γ. For the SISO channel this expression becomes

$$C\approx\ln\left(\frac{(1+\Gamma)^2}{\sqrt{1+2\Gamma+e^{\frac{\sigma^2}{\xi^2}}\Gamma^2}}\right). \tag{14}$$

III. Capacity with MRC and EGC

The instantaneous received SNR at the output of an M-branch maximum ratio combiner and equal gain combiner are, respectively, given by

$$\begin{cases}\gamma_{mrc}=\sum_{m=1}^{M}\gamma_m,\\ \gamma_{egc}=\frac{1}{M}(\sum_{m=1}^{M}\sqrt{\gamma_m})^2.\end{cases}$$

Exact expressions for the probability density functions of the RVs γ_{mrc} and γ_{egc} are unfortunately unknown. Notice however, that these RVs consist of a sum of log-normal RVs. We can therefore hinge on the log-normal approximation which states that the sum of log-normal Random Variables (RV) can be well approximated by another log-normal RV. Consequently γ_{mrc} and γ_{egc} are viewed as log-normal RVs thereby allowing us to use the previously established results for the SISO channel.

The logarithmic mean and logarithmic variance of the log-normal approximations of γ_{mrc} and γ_{egc} can be estimated by various methods. Here, we use the well known Fenton-Wilkinson (F-W) method because it provides a closed-form expression of the parameters of the log-normal RV and because of its simplicity. However, it should be noted that the F-W method performs badly for large standard deviations. In such cases, other methods should be used instead. Among these methods, we refer the interested reader to the method in [5] which seems to provide good results even for high standard deviations.

Without loss of generality, we assume in the following that the different diversity branches experience independent and identical fading[2], i.e., each branch has an average SNR equal to Γ and standard deviation equal to σ. We have therefore the following expressions

$$\begin{cases}\mu_{mrc}=\xi\ln(M\Gamma)-\frac{\sigma_{mrc}^2}{2\xi},\\ \sigma_{mrc}^2=\xi^2\ln\left(\frac{(M-1)+e^{\frac{\sigma^2}{\xi^2}}}{M}\right),\end{cases}$$

and

$$\begin{cases}\mu_{egc}=\xi\ln\left(\Gamma+\Gamma(M-1)e^{-\frac{\sigma^2}{4\xi^2}}\right)-\frac{\sigma_{egc}^2}{2\xi},\\ \sigma_{egc}^2=4\xi^2\ln\left(\frac{(M-1)+e^{\frac{\sigma^2}{4\xi^2}}}{M}\right).\end{cases}$$

The capacity for MRC and EGC is obtained therefore by substituting these values in (9) and in (13). For instance, for MRC, by using (13) we obtain

$$C_{mrc}\approx\ln\left(\frac{(1+M\Gamma)^2}{\sqrt{1+2M\Gamma+((M-1)+e^{\frac{\sigma^2}{\xi^2}})M\Gamma^2}}\right). \tag{15}$$

IV. Capacity in Interference Limited Environments

Here, we adopt the scenario considered in [7]. Namely, we analyze an interference-limited[3] environment in which the thermal noise is neglected. The desired user Signal to Interference Ratio (SIR) will be as follows

$$\gamma_{\text{SIR}}=\frac{\gamma_d}{\sum_{i=1}^{N_I}\gamma_i}, \tag{16}$$

where γ_d is the received power from the desired user, γ_i is the received power from the ith interferer and N_I denotes the number of interferers. We assume here for the sake of simplicity and without loss of generality that all the interferes are i.i.d. log-normal RVs. Using the log-normal approximation and following the steps in [7], γ_{SIR} is modeled as a log-normal RV with the following logarithmic variance

$$\sigma_{SIR}^2=\sigma_d^2+\xi^2\ln\left(\frac{(N_I-1)+e^{\frac{\sigma_I^2}{\xi^2}}}{N_I}\right), \tag{17}$$

where σ_d and σ_I are, respectively, the logarithmic variances of γ_d and $\gamma_i, 1\leq i\leq N_I$. The logarithmic mean of γ_{SIR} is

[2]Note that the extension to correlated fading is straightforward. For instance, we may use the method developed in [4] which extends the F-W method to the correlated scenario.

[3]We assume that all interfering users employ Gaussian code books, i.e., conditioned on the channel coefficients the interference term is gaussian distributed.

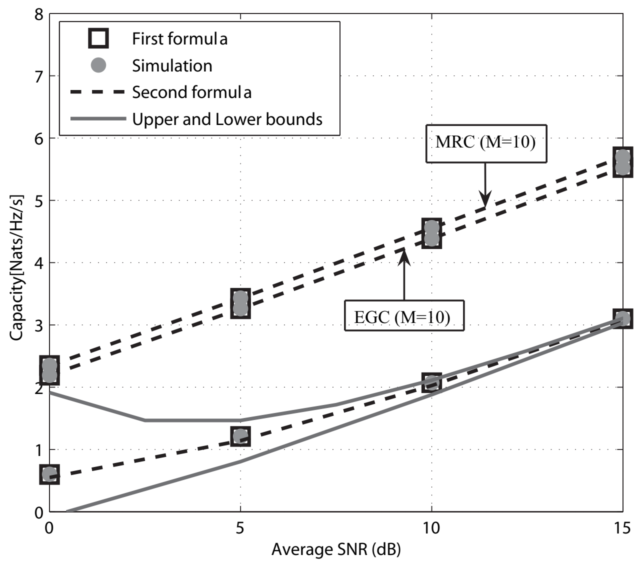

Fig. 1. Capacity in log-normal fading ($\sigma = 4$).

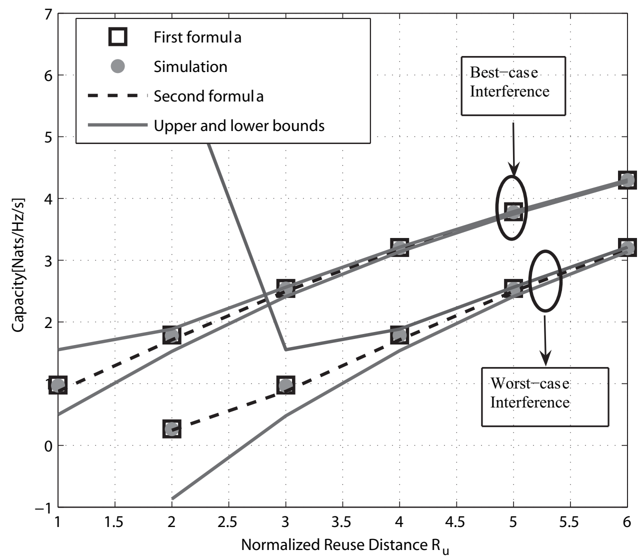

Fig. 2. Capacity in an interference-limited environment with $\sigma_d = \sigma_I = 4$ for the worst and the best-case interference ($a = b = 2$; $N_I = 6$; R=200; r=160; g=533).

given by [7]

$$\begin{aligned}\mu_{SIR}^{\pm} &= \xi \ln\left[\left(\frac{R(R_u \pm 1)}{r}\right)^a \left(\frac{g + R(R_u \pm 1)}{g + r}\right)^b\right] \\ &- \xi \ln(N_I) + \frac{\xi}{2} \ln\left(\frac{(N_I - 1)e^{-\frac{\sigma_I^2}{\xi^2}} + 1}{N_I}\right), \quad (18)\end{aligned}$$

where a is the basic path loss exponent ($a \approx 2$), b is the additional path loss exponent ($2 \leq b \leq 6$) and g is the break point of the path loss curve (in meters). R is the cell radius, r is the position of the desired user in the cell and R_u is the normalized reuse distance, i.e., $R_u = \frac{D}{R}$, where D denotes the reuse distance which is the distance between two base stations operating at the same frequencies. In (18), "+" is selected for the best-case interference while "-" is selected for the worst-case[4].

Finally, the capacity for the interference limited environment is obtained by plugging $\mu_{SIR}^{\pm}$ and σ_{SIR} in (9) and in (13).

V. Numerical Analysis

Fig. 1 depicts the capacity versus the average SNR in a log-normal fading channel. The logarithmic variance is set to $\sigma = 4$ dB. This figure depicts clearly the adequacy between the results obtained by simulations and those generated by the developed formulas. This figure depicts also the bounds that were developed in [7]. These bounds are loose for low SNR; the upper bound highly overestimates the capacity whereas the lower bound underestimates it.

Fig. 2 depicts the capacity versus the normalized reuse distance in an interference limited environment with 6 interferers for both the best and the worst-case interference. The standard deviation of the desired user and the different interferers are set to 4 dB. Here again, the developed approximations prove to be very accurate for a wide range of R_u while the bounds developed in [7] are loose for small normalized reuse distances.

VI. Conclusion

In this paper we have provided a very accurate approximation of the capacity of log-normal fading channels. We have also addressed the scenarios where diversity is used, namely, we have considered Maximum Ratio Combining and Equal Gain Combining. We have also assessed the capacity in an interference limited scenario. The analytical expressions provided match perfectly the capacity given by simulations.

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[4]The worst-case interference corresponds to the situation where the different interferers are at a distance $r_I^- = D - R$ from the desired user's Base-Station (BS) and the best-case interference corresponds to the configuration where the different interferers are at a distance $r_I^+ = D + R$ form this BS.