# On the Lower Performance Bounds for DOA Estimators from Linearly-Modulated Signals

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***Abstract*— In this paper, the problem of direction of arrival (DOA) estimation from linearly-modulated signals over AWGN channels is considered. We derive closed-form expressions for the inphase/quadrature Cramér-Rao lower bounds of the data-aided (DA) DOA estimates from any linearly-modulated signal corrupted by additive white circular complex Gaussian noise (AWCCGN). We consider the case of single-source signals impinging on multiple receiving antenna elements, commonly known as single input multiple output (SIMO) configurations. An analytical approach is conducted to compare the achievable performance in coherent estimation against noncoherent estimation over uniform linear array (ULA) and uniform circular array (UCA) configurations. It will be shown that the CRLBs that can be achieved over a ULA are lower than those that can be achievable over a UCA up to a given angular aperture whose expression is also derived in this paper. It will be shown also that ULAs exhibit lower CRLBs in coherent estimation than in noncoherent estimation and that the CRLBs hold, however, the same for UCAs in both estimation schemes.**

## I. Introduction

Direction of arrival (DOA) estimation of multiple plane waves impinging on an arbitrary array of antennas is an important task in array signal processing [1, 2]. Therfore, DOA estimation has attracted much interest during the last few decades since the knowledge of this parameter has been crucial for many applications. For example, in modern communication systems, estimating the DOAs of the desired users and those of the interference signals allows their extraction and cancellation, respectively, by beamforming technologies [3, 4]. Roughly speaking, DOA estimators can be mainly categorized into two categories: data-aided (DA) and non data-aided (NDA) estimators. DA DOA techniques assume perfect *a priori* knowledge of the transmitted symbols. NDA DOA techniques assume the latters to be completely unknown. DA estimators limit the throughput of the sytem. However, perfectly known sequences are usually transmitted for synchronization purposes. In this case, these known symbols can be used to design DA DOA estimation techniques without any addititional penalty. In this context, a number of high-resolution DOA estimation algorithms have been developed, including MUltiple SIgnal Classification (MUSIC) [5] and Estimation of Signal parameters via Rotational Invariance Technique (ESPRIT) [6]. Moreover, in the last few decades, there has been interest in developing other algorithms to improve the DOA estimation capability, especially the maximum likelihood (ML) approaches [7], the decoupled maximum likelihood (DEML) angle estimator [8] and the modified likelihood function approach [9].

In both DA and NDA cases, the variance of any DOA estimastor is usually compared to the Cramér-Rao lower bound (CRLB) which is a well-known fundamental performance limit that serves as a useful benchmark for practical estimators [10]. In the NDA mode, the CRLB is often numerically or empirically computed. But even when a closed-form expression can be obtained, it is usually complex and requires tedious algebraic manipulations. However, in the DA scenario, the CRLB is relatively easy to derive in closed-from expression. In this paper, we limit our analysis to the DA case for which we derive the closed-form expression for the CRLB of DA DOA unbiased estimates from any linearly-modulated signal. Actually, the DA CRLB was recently derived by Delams *et al.* in [11], but only for BPSK and QPSK transmissions. In this paper, however, we extend these results to any one-/two-dimensional constellation. Then, we focus our analysis on the comparison of the achievable performance between coherent and noncoherent estimations over the most used antenna configurations, namely the ULA and the UCA. We mention, however, that the analytical comparison carried out in this paper between ULA and UCA configurations, in the DA mode, is also valid in the NDA mode since this comparison is based on a geometrical factor which is the same for both DA and NDA scenarios. This can be seen in [11] for the special cases of BPSK and QPSK transmissions and for higher-order square QAM constellations in [12].

The rest of this paper is organized as follows. In section II, we introduce the system model that will be used throughout the article. In section III, explicit expressions for the DA Fisher information matrix (FIM) and the DA CRLB of the DOA estimates will be derived for any linearly-modulated signal. In section IV, the analytical comparison of the CRLB in coherent and noncoherent estimation schemes will be conducted over ULA and UCA configurations. Graphical representations and discussions will be given in section V and some concluding remarks will be drawn out in section VI.

Throughout this paper, matrices and vectors are represented by bold upper case and bold lower case letters, respectively.

## II. System Model

Consider a linearly-modulated (i.e., MPAM, MPSK, MQAM) signal impinging on an arbitrary array of $M$ antennas from a single source situated in the far-field region. We suppose that the received signal is corrupted by an additive white complex circular Gaussian noise (AWCCGN), spatially

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uncorrelated between antenna elements, with unknown noise power $\sigma^2$ over each antenna branch. Assuming a receiver with ideal time and frequency synchronization, the received signal at the output of the array matched filter can be modelled as a complex signal as follows:

$$\boldsymbol{y}(n) = S\ e^{j\phi}\ \boldsymbol{a}\ x(n) + \boldsymbol{w}(n), \quad n = 1, 2, \dots, N, \quad (1)$$

where, at time indices $\{n\}_{n=1}^{N}$, $\{x(n)\}_{n=1}^{N}$ are the $N$ transmitted independent and identically-distributed (iid) symbols and $\{\boldsymbol{y}(n)\}_{n=1}^{N}$ are the corresponding vectors of received samples on the antenna array. In this paper, we consider DA estimation scenarios and, therefore, the sequence $\{x(n)\}_{n=1}^{N}$ is considered as perfectly known to the receiver. $\boldsymbol{a}$ is the steering vector parametrized by the scalar DOA parameter $\theta$. The unknown parameters $S$ and $\phi$ stand for the constant channel gain and phase distortion, respectively. For any planar configuration of the receiving antenna array, the steering vector can be written as:

$$\boldsymbol{a} = \left[e^{j2\pi f_0(\theta)}, e^{j2\pi f_1(\theta)}, \dots, e^{j2\pi f_{M-1}(\theta)}\right]^T, \quad (2)$$

where the superscript $T$ stands for the transpose operator and $\{f_i(\theta)\}_{i=1,2,\dots,(M-1)}$ are transformations of the scalar DOA parameter $\theta$, which vary from one configuration to another. The steering vector verifies the property $||\boldsymbol{a}||^2 = M$, where $||.||$ returns the second norm of any vector. Moreover, we assume that the transmitted symbols $\{x(n)\}_{n=1,2,\dots,N}$ are independent from the noise components $\{\boldsymbol{w}(n)\}_{n=1,2,\dots,N}$ which are modelled by iid $M$-variate zero-mean complex circular Gaussian random vectors with independent real and imaginary parts and $\mathrm{E}\{\boldsymbol{w}(n)\boldsymbol{w}(n)^H\} = \sigma^2 \boldsymbol{I}_M$, where $\boldsymbol{I}_M$ refers to the $(M \times M)$ identity matrix. Furthermore, in order to derive standard CRLBs, the constellation energy is supposed to be normalized to one, i.e., $\mathrm{E}\{|x(n)|^2\} = 1$, where $\mathrm{E}\{.\}$ and $|.|$ return the expectation of any random variable and the modulus of any complex number, respectively.

The unknown parameters $\sigma$, $S$, $\phi$ and $\theta$ are more conveniently gathered in the following parameter vector:

$$\boldsymbol{\alpha} = [\sigma \quad S \quad \phi \quad \theta]^T, \quad (3)$$

The true SNR of the system is defined as follows:

$$\rho \;=\; \frac{S^2\ P}{\sigma^2},$$

where $P$ is the constellation power given for the DA and NDA modes, respectively, as $P = \frac{\sum_{n=1}^{N} |x(n)|^2}{N} \longrightarrow 1$ for $N \gg 1$ and $P = \mathrm{E}\{|x(n)|^2\} = 1$.

## III. Derivation of the I/Q Data-Aided FIM for DOA Estimates

In this section, we consider a general antenna array configuration. Since the transmitted symbols are assumed *a priori* known and the circular noise components are Gaussian distributed, then the received signal at the output of the array matched filter is circular Gaussian distributed. In fact, it can be shown that $P_{\mathrm{DA}}[\boldsymbol{y}(n);\boldsymbol{\alpha}] = P[\boldsymbol{y}(n)|x(n);\boldsymbol{\alpha}]$, the probability of the received vector $\boldsymbol{y}(n)$ parameterized by the unknown parameter vector $\boldsymbol{\alpha}$, is given by:

$$P_{\mathrm{DA}}[\boldsymbol{y}(n);\boldsymbol{\alpha}] \;=\; \frac{1}{\pi^M \sigma^{2M}} e^{-\frac{\left(\boldsymbol{y}(n) - S e^{j\phi} x(n)\boldsymbol{a}\right)^H \left(\boldsymbol{y}(n) - S e^{j\phi} x(n)\boldsymbol{a}\right)}{\sigma^2}}. \quad (4)$$

After some algebraic manipulations, the log-likelihood function of the received samples reduces to:

$$\ln(P_{\mathrm{DA}}[\boldsymbol{y}(n);\boldsymbol{\alpha}]) = -M\ln\left(\pi\sigma^2\right) - \frac{1}{\sigma^2}\left(||\boldsymbol{y}(n)||^2 - MS^2|x(n)|^2 + S\Re\{x(n)\}U(n) - S\Im\{x(n)\}V(n)\right), \quad (5)$$

with

$$U(n) \;=\; 2\ \Re\{e^{j\phi}\ \boldsymbol{y}(n)^H \boldsymbol{a}\}, \quad (6)$$
$$V(n) \;=\; 2\ \Im\{e^{j\phi}\ \boldsymbol{y}(n)^H \boldsymbol{a}\}. \quad (7)$$

In the above equations, $\Re\{.\}$ and $\Im\{.\}$ refer to the real and imaginary parts of any complex number, respectively.

In the only cases of BPSK and QPSK considered in [11], note that the DA CRLB for the DOA estimates is derived using the Slepian-Bangs formula [13, (B.3.25)]. In this paper, for any linearly-modulated signal, we will use another technique which is based on the evaluation of the expectation of the second partial derivatives of (5). Therefore, we need to evaluate beforehand the partial derivatives of the log-likelihood function and then derive the DA FIM associated with the four considered parameters $(\sigma, S, \phi, \theta)$. First, we partition the parameter vector into two parameter vectors $\boldsymbol{\alpha}^{(1)} = [S, \sigma^2]$ and $\boldsymbol{\alpha}^{(2)} = [\phi, \theta]$ and it will be later verified that:

$$\mathrm{E}\left\{\frac{\partial^2 \ln\left(P_{\mathrm{DA}}[\boldsymbol{y}(n);\boldsymbol{\alpha}]\right)}{\partial \boldsymbol{\alpha}_i^{(1)} \boldsymbol{\alpha}_l^{(2)}}\right\} = 0 \quad i, l = 1, 2, \quad (8)$$

where $\boldsymbol{\alpha}_i^{(m)}$ is the $i^{th}$ element of $\boldsymbol{\alpha}^{(m)}$ for $m = 1, 2$. Therefore, $\boldsymbol{\alpha}^{(1)}$ and $\boldsymbol{\alpha}^{(2)}$ are two decoupled parameter vectors and we are actually dealing with two seperate problems regarding the estimation of $S$ and $\sigma^2$, on the one hand, and the estimation of $\theta$ and $\phi$ on the other hand. This means that the DA FIM is block-diagonal structured as follows:

$$\boldsymbol{I}^{\mathrm{DA}}(\boldsymbol{\alpha}) = \begin{pmatrix} \boldsymbol{I}_1^{\mathrm{DA}}(\sigma, S) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_2^{\mathrm{DA}}(\phi, \theta) \end{pmatrix}, \quad (9)$$

where $\boldsymbol{0}$ is a $(2 \times 2)$ zero matrix. $\boldsymbol{I}_1^{\mathrm{DA}}(\sigma, S)$ is the DA FIM pertaining to the estimation of the channel amplitude and the noise power. $\boldsymbol{I}_2^{\mathrm{DA}}(\phi, \theta)$ is the DA FIM associated with the unknown channel phase and signal DOA. At this stage, as inferred from the block-diagonal structure of the global FIM, $\boldsymbol{I}^{\mathrm{DA}}(\boldsymbol{\alpha})$, we mention that whether the channel amplitude and the noise power are *a priori* known or not does not affect the ultimate accuracy on the DAO DA estimation. This is because, as $\boldsymbol{I}^{\mathrm{DA}}(\boldsymbol{\alpha})$ is block diagonal, the DOA CRLB (which corresponds to the fourth diagonal element of the inverse of $\boldsymbol{I}^{\mathrm{DA}}(\boldsymbol{\alpha})$ [15]) does not depend on the elements $\boldsymbol{I}_1^{\mathrm{DA}}(\sigma, S)$.

Moreover, since $\{\boldsymbol{y}(n)\}_{n=1}^{N}$ are i.i.d random vectors, the entries of the matrices $\boldsymbol{I}_1^{\mathrm{DA}}(\sigma,S)$ and $\boldsymbol{I}_2^{\mathrm{DA}}(\phi,\theta)$ are given by:

$$[\boldsymbol{I}_1^{\mathrm{DA}}(\sigma,S)]_{i,l}=-\sum_{n=1}^{N}\mathrm{E}\left\{\frac{\partial^2\ln(P_{\mathrm{DA}}[\boldsymbol{y}(n);\boldsymbol{\alpha}])}{\partial\boldsymbol{\alpha}_i^{(1)}\partial\boldsymbol{\alpha}_l^{(1)}}\right\}, i,l=1,2, \quad (10)$$

$$[\boldsymbol{I}_2^{\mathrm{DA}}(\phi,\theta)]_{i,l}=-\sum_{n=1}^{N}\mathrm{E}\left\{\frac{\partial^2\ln(P_{\mathrm{DA}}[\boldsymbol{y}(n);\boldsymbol{\alpha}])}{\partial\boldsymbol{\alpha}_i^{(2)}\partial\boldsymbol{\alpha}_l^{(2)}}\right\}, i,l=1,2. \quad (11)$$

In (10) and (11), the expectation $\mathrm{E}\{.\}$ is taken with respect to $\boldsymbol{y}(n)$. We show subsequently that for $i,l,m=1,2$, we have:

$$\mathrm{E}\left\{\frac{\partial^2\ln\left(P_{\mathrm{DA}}[\boldsymbol{y}(n);\boldsymbol{\alpha}]\right)}{\partial\boldsymbol{\alpha}_i^{(m)}\partial\boldsymbol{\alpha}_l^{(m)}}\right\} = C_{i,l,m}^1+C_{i,l,m}^2\,\mathrm{E}\left\{U(n)\right\}+ C_{i,l,m}^3\,\mathrm{E}\left\{V(n)\right\}, \quad (12)$$

where $C_{i,l,m}^1$, $C_{i,l,m}^2$ and $C_{i,l,m}^3$ are deterministic coefficients which depend on the unknown parameters $\{\boldsymbol{\alpha}_i^{(m)}\}_{i,m=1}^2$. To that end, we will only detail the derivation of $\mathrm{E}\{\frac{\partial^2\ln(P_{\mathrm{DA}}[\boldsymbol{y}(n);\boldsymbol{\alpha}])}{\partial\boldsymbol{\alpha}_2^{(2)}\partial\boldsymbol{\alpha}_1^{(2)}}\} = \mathrm{E}\left\{\frac{\partial^2\ln(P_{\mathrm{DA}}[\boldsymbol{y}(n);\boldsymbol{\alpha}])}{\partial\theta\partial\phi}\right\}$ and the other terms can then be easily derived in the same way. Indeed, we have:

$$\frac{\partial^2\ln\left(P_{\mathrm{DA}}[\boldsymbol{y}(n);\boldsymbol{\alpha}]\right)}{\partial\theta\partial\phi}=-\frac{S\Re\{x(n)\}\dot{V}(n)}{\sigma^2}-\frac{S\Im\{x(n)\}\dot{U}(n)}{\sigma^2},$$

where

$$\dot{U}(n) = \frac{\partial U(n)}{\partial\theta},\qquad \dot{V}(n) = \frac{\partial V(n)}{\partial\theta}.$$

Hence, we deduce:

$$\mathrm{E}\left\{\frac{\partial^2\ln\left(P_{\mathrm{DA}}[\boldsymbol{y}(n);\boldsymbol{\alpha}]\right)}{\partial\theta\partial\phi}\right\} = -\frac{S\Re\{x(n)\}}{\sigma^2}\mathrm{E}\left\{\dot{V}(n)\right\}- \frac{S\Im\{x(n)\}}{\sigma^2}\mathrm{E}\left\{\dot{U}(n)\right\},$$

and it can be shown that:

$$\mathrm{E}\left\{\dot{U}(n)\right\} = -\frac{j\dot{\boldsymbol{a}}^H\boldsymbol{a}}{M}\mathrm{E}\left\{V(n)\right\},\qquad \mathrm{E}\left\{\dot{V}(n)\right\} = \frac{j\dot{\boldsymbol{a}}^H\boldsymbol{a}}{M}\mathrm{E}\left\{U(n)\right\},$$

with

$$\dot{\boldsymbol{a}} = \frac{\partial\boldsymbol{a}}{\partial\theta}.$$

Consequently, we obtain:

$$\mathrm{E}\left\{\frac{\partial^2\ln\left(P_{\mathrm{DA}}[\boldsymbol{y}(n);\boldsymbol{\alpha}]\right)}{\partial\theta\partial\phi}\right\}=-\frac{j\dot{\boldsymbol{a}}^H\boldsymbol{a}S\ \Re\{x(n)\}}{M\sigma^2}\mathrm{E}\left\{U(n)\right\}+ \frac{j\dot{\boldsymbol{a}}^H\boldsymbol{a}S\ \Im\{x(n)\}}{M\sigma^2}\mathrm{E}\left\{V(n)\right\}. \quad (13)$$

We note from (13) that $\mathrm{E}\left\{\frac{\partial^2\ln(P_{\mathrm{DA}}[\boldsymbol{y}(n);\boldsymbol{\alpha}])}{\partial\theta\partial\phi}\right\}$ has the same form given in (12) with $C_{2,1,2}^1=0$.
In order to find an explicit expression for (13), we define the scalar random variable $\vartheta(n) = 2\ e^{j\phi}\ \boldsymbol{y}(n)^H\boldsymbol{a} = U(n)+jV(n)$ whose probability, in the DA case, is $P_{\mathrm{DA}}[\vartheta(n);\boldsymbol{\alpha}] = P[\vartheta(n)|x(n);\boldsymbol{\alpha}] = P_{\mathrm{DA}}[(U(n),V(n));\boldsymbol{\alpha}]$. Moreover, taking into account the hypothesis of the independence of $\Re\{x(n)\}$ and $\Im\{x(n)\}$, we show that $U(n)$ and $V(n)$ are independant and, therefore, we have:

$$P_{\mathrm{DA}}[(U(n),V(n));\boldsymbol{\alpha}] = P_{\mathrm{DA}}[U(n);\boldsymbol{\alpha}]P_{\mathrm{DA}}[V(n);\boldsymbol{\alpha}],$$

where it can be shown that:

$$P_{\mathrm{DA}}[U(n);\boldsymbol{\alpha}] = \frac{\exp\left\{-\frac{(U(n)-2MS\ \Re\{x(n)\})^2}{2(2M\sigma^2)}\right\}}{\sqrt{2\pi\,(2M\sigma^2)}}, \quad (14)$$

$$P_{\mathrm{DA}}[V(n);\boldsymbol{\alpha}] = \frac{\exp\left\{-\frac{(V(n)+2MS\ \Im\{x(n)\})^2}{2(2M\sigma^2)}\right\}}{\sqrt{2\pi\,(2M\sigma^2)}}. \quad (15)$$

We note from (14) and (15) that $U(n)$ and $V(n)$ are real Gaussian random variables with the same variance $2M\sigma^2$ and with means $\mathrm{E}\{U(n)\} = 2MS\ \Re\{x(n)\}$ and $\mathrm{E}\{V(n)\} = -2MS\ \Im\{x(n)\}$, respectively. Therefore, from (13), we obtain:

$$\mathrm{E}\left\{\frac{\partial^2\ln\left(P_{\mathrm{DA}}[\boldsymbol{y}(n);\boldsymbol{\alpha}]\right)}{\partial\theta\partial\phi}\right\}=-\frac{2(j\dot{\boldsymbol{a}}^H\boldsymbol{a})S^2}{\sigma^2}|x(n)|^2. \quad (16)$$

We note that in the special case of normalized-energy BPSK and QPSK constellations, we have $|x(n)| = 1$, for $n = 1,2,\ldots,N$ and hence $\mathrm{E}\left\{\frac{\partial^2\ln(P_{\mathrm{DA}}[\boldsymbol{y}(n);\boldsymbol{\alpha}])}{\partial\theta\partial\phi}\right\} = -\frac{2(j\dot{\boldsymbol{a}}^H\boldsymbol{a})S^2}{\sigma^2}$. Therefore, we have $\left[\boldsymbol{I}^{\mathrm{DA}}(\boldsymbol{\alpha})\right]_{3,4} = 2N\left(j\dot{\boldsymbol{a}}^H\boldsymbol{a}\right)\rho$, which is the same result previously obtained for these two special constellations in [11]. For any higher-order constellation, this result holds also as $\frac{S^2}{\sigma^2}\sum_{n=1}^{N}|x(n)|^2 = N\rho$. In fact, from (11), we have:

$$\left[\boldsymbol{I}^{\mathrm{DA}}(\boldsymbol{\alpha})\right]_{3,4} = \frac{2\left(j\dot{\boldsymbol{a}}^H\boldsymbol{a}\right)S^2}{\sigma^2}\sum_{n=1}^{N}|x(n)|^2 = 2N\left(j\dot{\boldsymbol{a}}^H\boldsymbol{a}\right)\rho.$$

Moreover, we note that the diagonal structure of the DA FIM is verified because $\mathrm{E}\left\{\frac{\partial^2\ln(P_{\mathrm{DA}}[\boldsymbol{y}(n);\boldsymbol{\alpha}])}{\partial\alpha_i\partial\alpha_l}\right\}_{i=1,2;\ l=3,4}$ involves the sum of two opposite terms. In fact, it can be verified that for $i,l=1,2$:

$$\mathrm{E}\left\{\frac{\partial^2\ln\left(P_{\mathrm{DA}}[\boldsymbol{y}(n);\boldsymbol{\alpha}]\right)}{\partial\boldsymbol{\alpha}_i^{(1)}\partial\boldsymbol{\alpha}_l^{(2)}}\right\}=2MSC_{i,l}\left(\Re\{x(n)\}\Im\{x(n)\}-\Re\{x(n)\}\Im\{x(n)\}\right) = 0.$$

Deriving the other terms of the DA FIM, in the same way, we obtain the following result:

$$\boldsymbol{I}^{\mathrm{DA}}(\boldsymbol{\alpha}) = N\begin{pmatrix} \frac{4M}{\sigma^2} & 0 & 0 & 0\\ 0 & \frac{2M}{\sigma^2} & 0 & 0\\ 0 & 0 & 2M\rho & 2\rho(j\dot{\boldsymbol{a}}^H\boldsymbol{a})\\ 0 & 0 & 2\rho(j\dot{\boldsymbol{a}}^H\boldsymbol{a}) & 2\rho||\dot{\boldsymbol{a}}||^2\end{pmatrix}. \quad (17)$$

Now that we have derived the expression of the FIM, in the DA mode, the considerd CRLBs can be easily deduced and, in the next section, we will compare these lower bounds between ULA and UCA configurations in both coherent and noncoherent estimations.

## IV. Comparison of the DA DOA CRLBs between ULA and UCA configurations

For a normalized constellation, inverting $\boldsymbol{I}^{\text{DA}}(\boldsymbol{\alpha})$, we deduce the expression for the DA CRLB in the presence of an unknown phase offset (i.e., noncoherent estimation)[1], $\text{CRLB}_{\text{DA}}^{\text{NCO}}$, of any linearly-modulated signal and for any planar antenna-array configuration as follows:

$$\text{CRLB}_{\text{DA}}^{\text{NCO}}(\theta) \;=\; \frac{1}{N\gamma\;\rho}, \tag{18}$$

where $\gamma$ is a purely "geometrical factor" defined as $\gamma = 2\dot{\boldsymbol{a}}^H\boldsymbol{\Pi}_{\boldsymbol{a}}^{\perp}\dot{\boldsymbol{a}}$ with $\boldsymbol{\Pi}_{\boldsymbol{a}}^{\perp} = \boldsymbol{I}_M - \boldsymbol{a}\boldsymbol{a}^H/M$. It can be verified that the expression given by (18) generalizes the expression recently derived in [11] from BPSK and QPSK signals to any higher-order one-/two-dimensional normalized constellation. Note also that this geometrical factor $\gamma$ appears as well in the expressions of the NDA DOA CRLBs of BPSK and QPSK signals [11] and higher-order square QAM signals [12]. Therefore, the analytical comparison between ULAs and UCAs that will be from now on conducted, using the expression of $\gamma$ in each configuration, applies also in the NDA mode.
To begin with, using (2), we show that $\gamma$ can be written as:

$$\begin{aligned}\gamma \;&=\; 2\left(||\dot{\boldsymbol{a}}||^2 - \frac{|\boldsymbol{a}^H\dot{\boldsymbol{a}}|^2}{M}\right), &(19)\\ &=\; 8\pi^2\left(\sum_{k=0}^{M-1}\dot{f}_k^2(\theta) - \frac{\left(\sum_{k=0}^{M-1}\dot{f}_k(\theta)\right)^2}{M}\right). &(20)\end{aligned}$$

We first show that, as it should be[2], $\gamma$ is stricly superior to zero. In fact, using the Cauchy-Schwarz inequality which states that for any linearly independant $(M\times 1)$-dimensional real vectors $\boldsymbol{x} = [x_1, x_2, \ldots, x_M]$ and $\boldsymbol{y}\,[y_1, y_2, \ldots, y_M]$, we have:

$$\left(\sum_{k=0}^{M-1}x_k y_k\right)^2 \;<\; ||\boldsymbol{x}||^2||\boldsymbol{y}||^2.$$

Replacing $x_k$ by $\dot{f}_k$ and $y_k$ by 1, we obtain:

$$\left(\sum_{k=0}^{M-1}\dot{f}_k(\theta)\right)^2 \;<\; M\sum_{k=0}^{M-1}\dot{f}_k(\theta)^2.$$

Then, from (20), it follows that $\gamma$ is stricly superior to zero.

In the absence of the phase offset or assuming that it was perfectly recovered (i.e., the parameter $\phi$ is supposed to be known, coherent estimation), the data-aided DOA CRLB, $\text{CRLB}_{\text{DA}}^{\text{CO}}$, can be obtained by simply inverting the second diagonal element of $\boldsymbol{I}_2^{\text{DA}}$ to yield:

$$\text{CRLB}_{\text{DA}}^{\text{CO}}(\theta) \;=\; \frac{1}{2\;N\;\rho\;||\dot{\boldsymbol{a}}||^2}. \tag{21}$$

We notice in (18) and (21) that the DA DOA CRLBs are inversely propotional to $N$. Therefore, they can be easily deduced for any observation interval size $N_2$ if they were already computed for a given window size $N_1$, by simply scaling with the factor $\frac{N_1}{N_2}$.

### A. Uniform linear array (ULA) configuration

Considering a uniform linear array (ULA), the steering vector is given by:

$$\boldsymbol{a} = \left[1, e^{j\pi\sin(\theta)}, e^{2j\pi\sin(\theta)}, \ldots, e^{j(M-1)\pi\sin(\theta)}\right]^T.$$

Consequently, from (19), $\gamma_{\text{ULA}}$ can be written as:

$$\begin{aligned}\gamma_{\text{ULA}} \;&=\; 2\pi^2\cos^2(\theta)\left(\sum_{k=1}^{M-1}k^2 - \frac{\left(\sum_{k=1}^{M-1}k\right)^2}{M}\right),\\ &=\; \pi^2\frac{M(M^2-1)}{6}\cos^2(\theta). &(22)\end{aligned}$$

Therefore, from (18), the expression of $\text{CRLB}_{\text{DA}}^{\text{NCO}}$ for a ULA configuration is:

$$\text{CRLB}_{\text{DA}}^{\text{NCO}}(\theta) \;=\; \frac{6}{NM(M^2-1)\pi^2\cos^2(\theta)\rho}. \tag{23}$$

On the other hand, for a ULA configuration, $||\dot{\boldsymbol{a}}||^2$ is given by:

$$\begin{aligned}||\dot{\boldsymbol{a}}||^2 \;&=\; \pi^2\cos^2(\theta)\sum_{k=1}^{M-1}k^2,\\ &=\; \pi^2\frac{M(M-1)(2M-1)}{6}\cos^2(\theta),\\ &=\; \frac{2M-1}{M+1}\,\gamma_{\text{ULA}}.\end{aligned}$$

Consequently, from (18) and (21), we deduce that for a ULA configuration:

$$\text{CRLB}_{\text{DA}}^{\text{CO}}(\theta) \;=\; \frac{3}{M(M-1)(2M-1)\pi^2 N\rho\cos^2(\theta)}, \tag{24}$$

(25)

Then, it can be easily deduced that:

$$\text{CRLB}_{\text{DA}}^{\text{CO}}(\theta) \;=\; \frac{M+1}{2(2M-1)}\;\text{CRLB}_{\text{DA}}^{\text{NCO}}(\theta). \tag{26}$$

We show that for any $M > 1$, $M+1 < 2(2M-1)$. Therefore, from (26), $\text{CRLB}_{\text{DA}}^{\text{CO}}(\theta)$ is lower than $\text{CRLB}_{\text{DA}}^{\text{NCO}}(\theta)$ for any antenna-array size $M > 1$. This is hardly surprising since in the absence of any phase offset , we have one less nuisance parameter which ultimately results in better achievable performance. It can also be seen from (22) that $\gamma_{\text{ULA}}$ is an even function of

[1] From now on, the superscripts CO and NCO will refer to coherent and noncoherent estimation scenarios, repectivley.

[2] This is because the CRLB is always positive and $\gamma$ sould not be equal to zero as it appears in the denomeraor.

$\theta$ and maximal for $\theta = 0$ since $\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. Therefore, from (23) and (24), the CRLB is an even function of $\theta$ and minimal for this value of $\theta$, thereby reflecting the symmetry of the ULA around the broadside axis.

### B. Uniform circular array (UCA) configuration

For a uniform circular array (UCA), the steering vector is given by [14]:

$$\boldsymbol{a} = \left[e^{\frac{j\pi\cos(\theta)}{2\sin(\pi/M)}}, e^{\frac{j\pi\cos(\theta-2\pi/M)}{2\sin(\pi/M)}}, \ldots, e^{\frac{j\pi\cos(\theta-2(M-1)\pi/M)}{2\sin(\pi/M)}}\right]^T.$$

Hence, $\gamma_{\text{UCA}}$ can be given by:

$$\gamma_{\text{UCA}} = \frac{\pi^2\left(\sum_{k=0}^{M-1}\sin^2\left(\theta-\frac{2k\pi}{M}\right) - \frac{\left(\sum_{k=1}^{M-1}\sin\left(\theta-\frac{2k\pi}{M}\right)\right)^2}{M}\right)}{2\sin^2\left(\frac{\pi}{M}\right)}.$$

Using the identities $\sin\left(\theta-\frac{2k\pi}{M}\right) = \Im\left\{e^{j\left(\theta-\frac{2k\pi}{M}\right)}\right\}$, $\cos\left(\theta-\frac{2k\pi}{M}\right) = \Re\left\{e^{j\left(\theta-\frac{2k\pi}{M}\right)}\right\}$ and $\sum_{k=0}^{M-1} e^{\frac{2jk\pi}{M}} = 0$, we show that $\gamma_{\text{UCA}}$ reduces simply to:

$$\gamma_{\text{UCA}} = \frac{M\pi^2}{4\sin^2\left(\frac{\pi}{M}\right)}, \tag{27}$$

which means that the CRLB for a UCA is the same for any $\theta$, thereby reflecting the circular symmetry of the UCA. Consequently, from (18), the expression of $\text{CRLB}_{\text{DA}}^{\text{NCO}}$ for a UCA configuration is given by:

$$\text{CRLB}_{\text{DA}}^{\text{NCO}}(\theta) = \frac{4\sin^2\left(\frac{\pi}{M}\right)}{NM\pi^2\rho}. \tag{28}$$

On the other hand, for a UCA configuration, $||\dot{\boldsymbol{a}}||^2$ is given by:

$$\begin{aligned} ||\dot{\boldsymbol{a}}||^2 &= \frac{\pi^2}{4\sin^2\left(\frac{\pi}{M}\right)}\sum_{k=0}^{M-1}\sin^2\left(\theta-\frac{2k\pi}{M}\right), \\ &= \frac{M\pi^2}{8\sin^2\left(\frac{\pi}{M}\right)}, \\ &= \frac{\gamma_{\text{UCA}}}{2}. \end{aligned}$$

Consequently, from (18) and (21), we deduce that, for a UCA configuration, the DA CRLB is the same for both coherent and noncoherent estimations and we have:

$$\text{CRLB}_{\text{DA}}^{\text{CO}}(\theta) = \text{CRLB}_{\text{DA}}^{\text{NCO}}(\theta) = \frac{4\sin^2\left(\frac{\pi}{M}\right)}{N\ \rho M\pi^2}. \tag{29}$$

We conclude from (29) that, for a DA scenario, the knowledge of the phase offset for a UCA configuration does not bring any additional CRLB performance gain, contrarily to the ULA configuration.

## V. Graphical Representations

In this section, to illustrate the behavior of the derived lower bounds, we provide a graphical representation of the DA CRLBs of both coherent and noncoherent DOA estimations as a function of the SNR for a fixed DOA parameter $\theta = 0$.

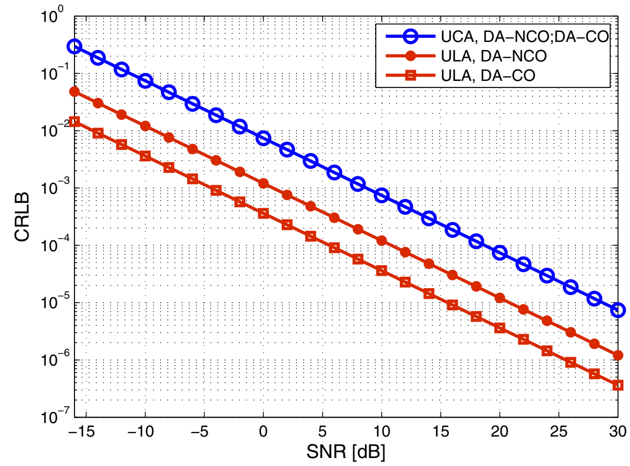


Fig. 1. DA CRLBs for the DOA estimates with ULA and UCA configurations, $p = 2$, $N = 1$, $M = 8$ .

In Fig. 1, we compare the DOA CRLBs for both the ULA and UCA antenna-array configurations for $M = 8$ receiving antenna elements. We see from this figure that, as shown analytically, the achievable performance on the DOA estimates depends on the geometrical configuration of the receiving antenna array. In fact, for the considered value of $\theta = 0$, a ULA offers better estimation performance than UCA. Actually, for a fixed number of antennas $M$, we prove that the CRLBs obtained for a ULA configuration are lower than the CRLBs obtained with a UCA for any value of $\theta \in [-\theta_c, \theta_c]$ with $\theta_c = \arccos\left(\sqrt{\frac{3}{2\sin^2\left(\frac{\pi}{M}\right)(M-1)(M+1)}}\right)$. UCAs are, however, more used in practice. Indeed, the major avantage of UCAs is their 360 degrees azimuthal coverage which is necessary in many applications such as wireless communications, radar and sonar and their almost invariant directional pattern. This is in strong contrast with the widely studied ULA that only covers 180 degrees.

## VI. Conclusion

In this paper, we developed analytical expressions for the CRLBs of the DA DOA CRLBs for any linearly-modulated signal as a function of the true SNR values. The received samples are assumed to be corrupted by additive white circular complex Gaussian noise. We proved that the achievable performance on the DA DOA estimates depends on the geometrical configuration of the antenna array. In fact, with a ULA, we showed that the CRLBs are even functions of the DOA that reach their minima at broadside direction due to the ULA's axial symmetry around it. Moreover, the ULA CRLBs in coherent estimation are lower than those obtained in noncoherent estimation. With a UCA, we showed however that the CRLBs no longer depend on the DOA due to its circular symmetry and that the CRLB in both coherent and noncoherent estimations are equal. We also showed that, for a given antenna-array size $M$, ULAs exhibit lower CRLBs than UCAs up to an angle aperture that depends on $M$.

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