

Closed-Form Expressions for the Exact Cramér-Rao Bound for Parameter Estimation of Arbitrary Square QAM-Modulated Signals

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Abstract— In this paper, we derive analytical expressions for the exact Cramér-Rao lower bounds (CRLBs) for the joint estimation of the carrier frequency, the carrier phase, the noise power and the signal amplitude of square quadrature amplitude (QAM)-modulated signals. The channel is assumed to be slowly time-varying so that it can be assumed constant over the observation interval. The signal is assumed to be corrupted by additive white Gaussian noise (AWGN). The closed-form expressions for the corresponding modified Cramér-Rao lower bounds (MCRLBs) are also derived in this paper.

I. INTRODUCTION

Quadrature amplitude modulation (QAM) is a relevant technique to convey data in modern digital communications. In practice, coherent receivers must compensate the unknown phase and frequency offset introduced by the channel using synchronization techniques. Moreover, recovering the sample functions from the received waveforms often requires accurate estimates of certain signal parameters such as the carrier frequency, the carrier phase, the noise power and the signal amplitude. Establishing bounds on the ultimate accuracy that can be achieved in the estimation of these parameters is an important goal since it provides benchmarks for evaluating the performance of actual estimators.

Tools to approach this problem are available from the parameter estimation theory [1] in the form of the Cramér-Rao lower bound (CRLB) which is a lower bound on the variance of any unbiased estimator. The CRLB is known, among many other bounds that have been introduced in the literature [1], to be the easiest to determine. This bound is often used in signal processing when dealing with deterministic unknown parameter estimation.

Actually, the CRLBs for the Non Data Aided phase and frequency estimates were introduced in [2] assuming the channel gain and the noise power to be perfectly known to the receiver. But, depending on the SNR region, these CRLBs were numerically or empirically computed and no analytical expressions for these bounds were derived there [2]. Years later, closed-form expressions for these CRLBs assuming the channel gain and the noise power to be completely unknown, have been derived ultimately in [3], but only in the special cases of BPSK and QPSK signals.

Because the derivation of the stochastic CRLB was thought to be prohibitive in [4], the authors considered arbitrary deterministic signals corrupted by circular complex Gaussian noise.

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Hence, the likelihood function of the received samples remains Gaussian and the associated deterministic CRLB was easily derived. But, contrarily to the deterministic CRLB which is known to be not achievable in the general case, the stochastic CRLB can be achieved asymptotically (in the number of measurements) by several high resolution methods, such as the stochastic maximum likelihood (ML) estimator.

In this paper, our concern is to derive the general closed-form expressions of CRLBs for parameter estimation of any square QAM constellation. The modified CRLBs (MCRLBs) [5], which are looser bounds than the true CRLBs, but much simpler to derive, are also evaluated for the different considered unknown deterministic parameters.

The rest of this paper is organized as follows. In section II, we will introduce the system model that will be used throughout the article. In section III, we will derive the closed-form expressions for the different Fisher information matrices (FIMs) and the considered CRLBs. Some graphical representations of the newly derived CRLBs will be presented in section IV. Finally, some concluding remarks will be drawn out in section V.

II. SYSTEM MODEL

Consider a traditional digital communication system broadcasting and receiving any square QAM-modulated signal. The channel is supposed to be of a constant gain coefficient S over the observation interval. We assume that we receive an AWGN-corrupted signal where the noise power is σ^2 . Assuming an ideal receiver with perfect synchronization, the received signal at the output of the matched filter can be modelled as a complex signal as follows :

$$y(k) = S a(k) e^{j2\pi k\nu} e^{j\phi} + w(k), \quad k = k_0, \dots, k_0+K-1, \quad (1)$$

where, at time index k , $a(k)$ is the transmitted symbol and $y(k)$ is the corresponding received sample. The noise component $w(k)$ is modelled by a zero-mean Gaussian random variable with independent real and imaginary parts, each of variance $\sigma^2/2$. K is the number of the received samples in the observation interval.

Moreover, the transmitted symbols are assumed to be independent and identically distributed (iid) and drawn from any M -ary square QAM constellation, i.e. $M = 2^{2p}$ ($p = 1, 2, 3 \dots$). ϕ , ν and S are the deterministic unknown parameters representing, respectively, the carrier phase, the carrier frequency and the signal amplitude. In order to derive standard CRLBs, the square QAM constellation energy is supposed to be normalized to one, i.e., $E\{|a(k)|^2\} = 1$ where $E\{\cdot\}$ and $|\cdot|$ refer to

the expectation of any random variable and the module of any complex number, respectively. We define the following unknown parameter vector :

$$\boldsymbol{\theta} = [\nu \ \phi \ S \ \sigma^2]^T, \quad (2)$$

where the superscript T denotes the transpose operator. The Signal-to-Noise Ratio (SNR) is defined as :

$$\rho = \frac{S^2}{\sigma^2}. \quad (3)$$

III. DERIVATION OF THE PARAMETERS' CRLBs

In this section, we will derive the closed-form expressions for the CRLBs of the carrier phase ϕ , the carrier frequency ν , the noise signal σ^2 and the signal amplitude S when the transmitted signal is square QAM-modulated and AWGN-corrupted. As shown in [6], the CRLB for parameter vector estimation is given by

$$\text{CRLB}(\boldsymbol{\theta}) = \mathbf{I}^{-1}(\boldsymbol{\theta}), \quad (4)$$

where $\mathbf{I}(\boldsymbol{\theta})$ is the Fisher information matrix (FIM) defined as

$$[\mathbf{I}(\boldsymbol{\theta})]_{ij} = -\text{E} \left\{ \frac{\partial^2 \ln(P[\mathbf{y}; \boldsymbol{\theta}])}{\partial \theta_i \partial \theta_j} \right\}, \quad (5)$$

where $\mathbf{y} = [y(k_0), y(k_0 + 1), \dots, y(k_0 + K - 1)]^T$, θ_i is the i^{th} element of $\boldsymbol{\theta}$ and $P[\mathbf{y}; \boldsymbol{\theta}]$ is the probability density function of \mathbf{y} parameterized by $\boldsymbol{\theta}$. The expectation $\text{E}\{\cdot\}$ is taken with respect to \mathbf{y} .

Usually, the derivation of the CRLB involves tedious algebraic manipulations. These mainly consist in the derivation of the FIM elements. We now give the major final results.

In fact, under the assumptions made so far and for any M -ary QAM constellation (i.e. $M = 2^p$ for arbitrary integer $p \geq 2$), it can be shown that the probability $P[y(k); \boldsymbol{\theta}]$ of the received sample $y(k)$ parameterized by $\boldsymbol{\theta}$ is given by :

$$P[y(k); \boldsymbol{\theta}] = \frac{1}{M\pi\sigma^2} \exp \left\{ -\frac{I(k)^2 + Q(k)^2}{\sigma^2} \right\} D_{\boldsymbol{\theta}}(k), \quad (6)$$

where $I(k)$ and $Q(k)$ are, respectively, the inphase/real and quadrature/imaginary component/part of the corresponding received sample $y(k)$, which means $y(k) = I(k) + jQ(k)$, and $D_{\boldsymbol{\theta}}(k)$ is given by :

$$D_{\boldsymbol{\theta}}(k) = \sum_{c_i \in \mathcal{C}} \exp \left\{ -\frac{S^2 |c_i|^2}{\sigma^2} \right\} \times \exp \left\{ \frac{2S \Re\{y(k)^* e^{j(2\pi k\nu + \phi)} c_i\}}{\sigma^2} \right\}, \quad (7)$$

where \mathcal{C} is the constellation alphabet and $\Re\{\cdot\}$ and the superscript $*$ return the real part and the conjugate of any complex number, respectively.

As mentioned previously, to reduce the computational complexity, the number of the nuisance parameters was reduced by assuming the signal amplitude and noise power to be perfectly known. Moreover, due to the complexity of the likelihood function, for a general linearly-modulated signal, the CRLBs for the joint phase and frequency estimation were numerically or empirically computed, depending on the SNR region. In fact, at low SNR values, the authors resorted to numerical double integration and at moderate and high SNRs Monte-Carlo evaluation (MCE) was used.

However, in this paper, considering only square QAM constellations, we are able to derive analytical expressions for the CRLBs as a function of the true SNR values ρ . Moreover, contrarily to [2], we assume the noise power and the signal amplitude to be deterministic unknown parameters and as recently done in [3] in the special cases of BPSK and QPSK signals. In fact, the major advantage offered by these square constellations is that $P[y(k); \boldsymbol{\theta}]$ can be factorized, making it possible to obtain simpler analytical expressions, as a function of the true SNR ρ , for the FIM elements given by (5).

Indeed, when $M = 2^{2p}$ for any $p \geq 1$, we have $\mathcal{C} = \{\pm(2i - 1)d_p \pm j(2l - 1)d_p\}_{i,l=1,2,\dots,2^{p-1}}$ where $j^2 = -1$ and $2d_p$ is the intersymbol distance in the I/Q plane. Therefore, we show that $D_{\boldsymbol{\theta}}$ can be written as follows :

$$D_{\boldsymbol{\theta}}(k) = 4F_{\boldsymbol{\theta}}(U(k))F_{\boldsymbol{\theta}}(V(k)), \quad (8)$$

where

$$F_{\boldsymbol{\theta}}(t) = \sum_{i=1}^{2^{p-1}} \exp \left\{ -\frac{S^2(2i-1)^2 d_p^2}{\sigma^2} \right\} \cosh \left(\frac{2(2i-1)d_p S t}{\sigma^2} \right), \quad (9)$$

$$U(k) = I(k) \cos(2\pi k\nu + \phi) + Q(k) \sin(2\pi k\nu + \phi), \quad (10)$$

$$V(k) = I(k) \sin(2\pi k\nu + \phi) - Q(k) \cos(2\pi k\nu + \phi). \quad (11)$$

For a normalized square QAM constellation, d_p is computed using the following assumption :

$$\frac{\sum_{l=1}^{2^{2p}} |c_l|^2}{2^{2p}} = 1, \quad (12)$$

which yields the following result :

$$d_p = \frac{2^{p-1}}{\sqrt{2^p \sum_{l=1}^{2^{2p-1}} (2l-1)^2}}. \quad (13)$$

Moreover, since the transmitted symbols are assumed to be iid, then the corresponding AWGN-corrupted received samples are independent and the probability of the received vector $\mathbf{y} = [y(k_0), y(k_0 + 1), \dots, y(k_0 + K - 1)]$ parameterized by $\boldsymbol{\theta}$ is given by :

$$P[\mathbf{y}; \boldsymbol{\theta}] = \left(\frac{4}{M\pi\sigma^2} \right)^K \exp \left\{ -\sum_{k=k_0}^{k_0+K-1} \frac{I(k)^2 + Q(k)^2}{\sigma^2} \right\} \times \prod_{k=k_0}^{k_0+K-1} F_{\boldsymbol{\theta}}(U(k))F_{\boldsymbol{\theta}}(V(k)). \quad (14)$$

Finally, the log-likelihood function of the received samples is given by :

$$\ln(P[\mathbf{y}; \boldsymbol{\theta}]) = K \ln \left(\frac{4}{M\pi\sigma^2} \right) - \sum_{k=k_0}^{k_0+K-1} \frac{I(k)^2 + Q(k)^2}{\sigma^2} + \sum_{k=k_0}^{k_0+K-1} \ln(F_{\boldsymbol{\theta}}(U(k))) + \sum_{k=k_0}^{k_0+K-1} \ln(F_{\boldsymbol{\theta}}(V(k))). \quad (15)$$

As it can be seen from (15), due to the factorization of the received samples probability, the log-likelihood function involves the sum of two analogous terms. This reduces the complexity of the derivation of the second partial derivatives and their expected values. Moreover, as in [3], we partition

the parameter vector $\boldsymbol{\theta} = [\nu \ \phi \ S \ \sigma^2]^T$ into two decoupled parameter vectors $[\nu \ \phi]^T$ and $[S \ \sigma^2]^T$. In Appendix, we show that the expected values of the second derivatives with respect to an element from $\boldsymbol{\theta}^{(1)}$ and an element from $\boldsymbol{\theta}^{(2)}$ are all equal to zero. This can be formally written in the following succinct form :

$$\mathbb{E} \left\{ \frac{\partial^2 \ln(P[\mathbf{y}; \boldsymbol{\theta}])}{\partial \theta_i^{(1)} \partial \theta_j^{(2)}} \right\} = 0, \quad i, j = 1, 2. \quad (16)$$

Thus, the FIM associated with the square QAM modulated-signals is given by the following diagonal matrix :

$$\mathbf{I}(\boldsymbol{\theta}) = \begin{pmatrix} \mathbf{I}^{(1)} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}^{(2)} \end{pmatrix}, \quad (17)$$

where $\mathbf{0}$ is a (2×2) zero matrix, $\mathbf{I}^{(2)}$ is derived in [7, eq.(19)] and $\mathbf{I}^{(1)}$ is given after tedious algebraic manipulations as :

$$\mathbf{I}^{(1)} = 2\rho \left(\rho - \frac{(1+\rho)}{A_2} F(\rho) \right) \times \begin{pmatrix} (2\pi)^2 \sum_{k=k_0}^{k_0+K-1} k^2 & 2\pi \sum_{k=k_0}^{k_0+K-1} k \\ 2\pi \sum_{k=k_0}^{k_0+K-1} k & K \end{pmatrix}, \quad (18)$$

where A_2 and $F(\rho)$ are given by :

$$A_2 = \sum_{l=1}^{2^{p-1}} (2l-1)^2, \quad (19)$$

$$F(\rho) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f_\rho(t) e^{-\frac{t^2}{2}} dt, \quad (20)$$

and

$$f_\rho(t) = \frac{\left[\sum_{i=1}^{2^{p-1}} (2i-1) e^{-\rho(2i-1)^2 d_p^2} \sinh(\sqrt{2\rho}(2i-1)d_p t) \right]^2}{\sum_{i=1}^{2^{p-1}} e^{-\rho(2i-1)^2 d_p^2} \cosh(\sqrt{2\rho}(2i-1)d_p t)}. \quad (21)$$

Next, we will derive the modified CRLBs (MCRLBs) [5-8] which provide good approximations for the true bound for MPSK/QAM-modulated signals at high SNR values. This bound on the variance of any unbiased estimator is given by :

$$\text{MCRLB}(\boldsymbol{\theta}) = \frac{1}{\mathbb{E}_{\mathbf{a}} \left\{ \mathbb{E}_{\mathbf{y}|\mathbf{a}} \left\{ -\frac{\partial^2 \ln(P[\mathbf{y}|\mathbf{a}; \boldsymbol{\theta}])}{\partial \boldsymbol{\theta}^2} \right\} \right\}}, \quad (22)$$

where $\mathbb{E}_{\mathbf{y}|\mathbf{a}}\{\cdot\}$ (resp. and $\mathbb{E}_{\mathbf{a}}\{\cdot\}$) denotes expectation with respect to $P[\mathbf{y}|\mathbf{a}; \boldsymbol{\theta}]$ (resp. $P[\mathbf{a}]$).

We will only derive the MCRLB for the carrier phase and the MCRLBs for the remaining parameters follow similarly. To do so, we define \mathcal{M} as the set of the different $(K \times 1)$ vectors whose elements are chosen among $\{|c_1|^2, |c_2|^2, \dots, |c_M|^2\}$ where c_i is the i^{th} symbol in the constellation alphabet. Then, after tedious algebraic manipulations, we obtain the following result :

$$\mathbb{E}_{\mathbf{y}|\mathbf{a}} \left\{ -\frac{\partial^2 \ln(P[\mathbf{y}; \boldsymbol{\theta}])}{\partial \phi^2} \right\} = 2\rho \sum_{k=k_0}^{k_0+K-1} |a(k)|^2 \quad (23)$$

$$= 2\rho \mathbf{1}^T \mathbf{u}, \quad (24)$$

where $\mathbf{1}$ and \mathbf{u} are the $(K \times 1)$ -dimensional vectors given by :

$$\mathbf{1} = [1, 1, \dots, 1]^T, \quad (25)$$

$$\mathbf{u} = [|a_{k_0}|^2, |a_{k_0+1}|^2, \dots, |a_{k_0+K-1}|^2]^T. \quad (26)$$

Recall that $a(k)$, for $k = k_0 \dots, k_0 + K - 1$, is a uniform random variable taking values in $\mathcal{C} = \{c_1, c_2, \dots, c_M\}$ with equal probabilities $\frac{1}{M}$ and $\{a(k)\}_{k=k_0 \dots, k_0+K-1}$ are assumed to be iid. Therefore, \mathbf{u} is an K variate uniform random variable taking values in \mathcal{M} with equal probabilities $(\frac{1}{M})^K$.

Consequently, we have :

$$\text{MCRLB}(\phi) = \frac{1}{2\rho \mathbf{1}^T \mathbb{E}_{\mathbf{a}} \{\mathbf{u}\}}, \quad (27)$$

$$= \frac{1}{2\rho \mathbf{1}^T \sum_{\mathbf{u} \in \mathcal{M}} (\frac{1}{M})^K \mathbf{u}}. \quad (28)$$

Besides, we show that :

$$\sum_{\mathbf{u} \in \mathcal{M}} \mathbf{u} = M^K \mathbf{1}, \quad (29)$$

Hence, we get the following result :

$$\text{MCRLB}(\phi) = \frac{1}{2K\rho}. \quad (30)$$

Equivalent derivations of the MCRLBs of the carrier frequency ν , the amplitude S and the noise power σ^2 yield the following results :

$$\text{MCRLB}(\nu) = \frac{1}{2(2\pi)^2 \rho \sum_{k=k_0}^{k_0+K-1} k^2}, \quad (31)$$

$$\text{MCRLB}(S) = \frac{S^2}{2K\rho}, \quad (32)$$

$$\text{MCRLB}(\sigma^2) = \frac{\sigma^4}{K}. \quad (33)$$

Moreover, when the joint estimator is analyzed relative to the middle of the signal vector (i.e., $k_0 = -(K-1)/2$), the FIM becomes diagonal and the frequency and phase parameters are decoupled. In this case, inverting the FIM given by (18), we get the closed-form expressions for CRLBs of the joint estimation of the parameters ν and ϕ as follows :

$$\text{CRLB}(\nu) = \frac{-6}{(2\pi)^2 K(K^2-1)\rho \left(\rho - \frac{(1+\rho)F(\rho)}{A_2} \right)}, \quad (34)$$

$$= \text{MCRLB}(\nu) \left(\frac{-1}{\rho - \frac{(1+\rho)}{A_2} F(\rho)} \right). \quad (35)$$

$$\text{CRLB}(\phi) = \frac{-1}{2K\rho \left(\rho - \frac{(1+\rho)F(\rho)}{A_2} \right)}, \quad (36)$$

$$= \text{MCRLB}(\phi) \left(\frac{-1}{\rho - \frac{(1+\rho)}{A_2} F(\rho)} \right). \quad (37)$$

Now, using the newly derived expressions in (32) and (33), and inverting $\mathbf{I}^{(2)}$, which was recently derived in [7], we establish the following relationship between the CRLBs and

the MCRLBs for the noise power σ^2 and the signal amplitude S as follows :

$$\text{CRLB}(S) = \frac{2^{p-1} S^2}{K \rho} \frac{I_{2,2}^{(2)}}{\Gamma(\rho) I_{2,2} - 2\rho(A_2 d_p^2 - \Delta(\rho))^2}, \quad (38)$$

$$= \text{MCRLB}(S) \frac{2^{p-2} I_{2,2}^{(2)}}{\Gamma(\rho) I_{2,2} - 2\rho(A_2 d_p^2 - \Delta(\rho))^2}, \quad (39)$$

$$\text{CRLB}(\sigma^2) = \frac{2^{p-2} \sigma^4}{K} \frac{\Gamma(\rho)}{\Gamma(\rho) I_{2,2} - 2\rho(A_2 d_p^2 - \Delta(\rho))^2}, \quad (40)$$

$$= \text{MCRLB}(\sigma^2) \frac{2^{p-2} \Gamma(\rho)}{\Gamma(\rho) I_{2,2} - 2\rho(A_2 d_p^2 - \Delta(\rho))^2}, \quad (41)$$

where A_4 , $I_{2,2}^{(2)}$, $\Gamma(\rho)$, $\Lambda(\rho)$, $\Delta(\rho)$, $\gamma(\rho)$, $\delta(\rho)$ and $\lambda(\rho)$ are given by :

$$I_{2,2}^{(2)} = 2^{p-2} + \frac{S^2}{\sigma^2} [A_2 d_p^2 + \Lambda(\rho)] - \rho(4A_2 d_p^2 + \rho A_4), \quad (42)$$

$$A_4 = \sum_{l=1}^{2^{p-1}} (2l-1)^4 d_p^4, \quad (43)$$

$$\Gamma(\rho) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\gamma_\rho^2(t)}{\delta_\rho(t)} e^{-\frac{t^2}{2}} dt, \quad (44)$$

$$\Lambda(\rho) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\lambda_\rho^2(t)}{\delta_\rho(t)} e^{-\frac{t^2}{2}} dt, \quad (45)$$

$$\Delta(\rho) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\gamma_\rho(t) \lambda_\rho(t)}{\delta_\rho(t)} e^{-\frac{t^2}{2}} dt, \quad (46)$$

$$\begin{aligned} \gamma_\rho(t) = & \sum_{i=1}^{2^{p-1}} e^{-(2i-1)^2 d_p^2 \rho} \\ & \left[(2i-1) d_p t \sinh \left((2i-1) d_p \sqrt{2\rho} t \right) - \right. \\ & \left. (2i-1)^2 d_p^2 \sqrt{2\rho} \cosh \left((2i-1) d_p \sqrt{2\rho} t \right) \right], \quad (47) \end{aligned}$$

$$\begin{aligned} \lambda_\rho(t) = & \sum_{i=1}^{2^{p-1}} e^{-(2i-1)^2 d_p^2 \rho} \\ & \left[(2i-1) d_p t \sinh \left((2i-1) d_p \sqrt{2\rho} t \right) - \right. \\ & \left. (2i-1)^2 d_p^2 \sqrt{\frac{\rho}{2}} \cosh \left((2i-1) d_p \sqrt{2\rho} t \right) \right], \quad (48) \end{aligned}$$

$$\delta_\rho(t) = \sum_{i=1}^{2^{p-1}} e^{-(2i-1)^2 d_p^2 \rho} \cosh \left((2i-1) d_p \sqrt{2\rho} t \right). \quad (49)$$

As it can be seen from (36), for any square QAM constellation, the CRLBs of the phase (respectively of the frequency) do not depend on the phase ϕ introduced by the channel, as shown earlier in [3] for BPSK and QPSK modulated signals only. Finally, it is worth noting that the analytical expression for the CRLBs as a function of the true SNR, established in (35), (37) and (39), generalize the elegant CRLB expression derived in [3] for QPSK constellations to higher-order square QAM modulations. In fact, it can be verified that the closed-form expressions of the FIM derived in [3, (3)...(6)] for QPSK signals correspond to the special case of $p = 1$ in the general expressions derived in this paper¹.

1. It should be noted that, in this paper, to refer to the noise power, we considered the parameter σ^2 instead of σ (as done in [3]).

IV. GRAPHICAL REPRESENTATIONS

In this section, we include some graphical representations of the lower bounds given by (35) and (37) for different modulation orders. It should be mentioned that, as seen from (35) and (37), the ratio of the CRLB to the MCRLB is the same for the phase and the frequency estimated from the same K received samples. Moreover, as shown in Fig. (1), we see

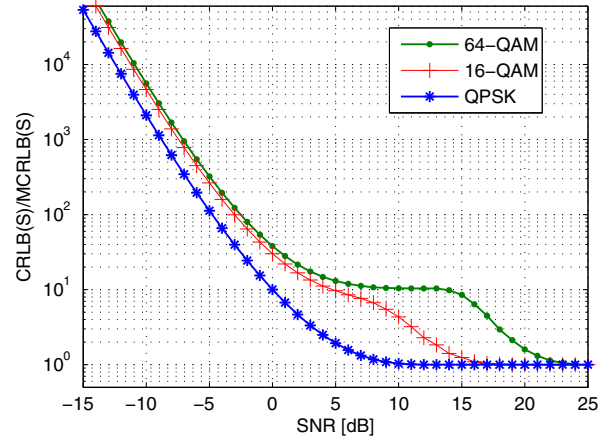


Fig. 1. $\text{CRLB}(\nu)/\text{MCRLB}(\nu) = \text{CRLB}(\phi)/\text{MCRLB}(\phi)$ at different SNR values.

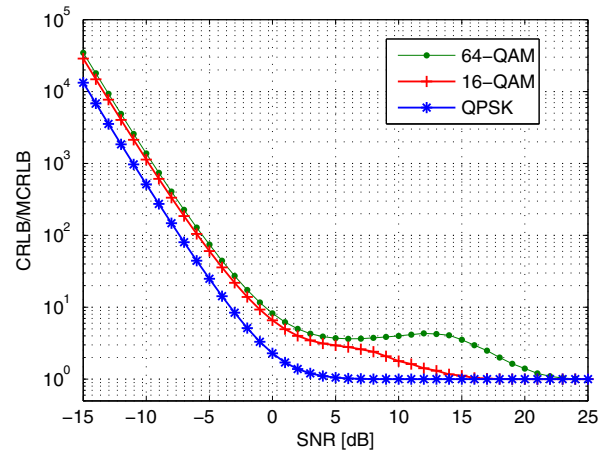


Fig. 2. $\text{CRLB}(S)/\text{MCRLB}(S)$ at different SNR values.

a good agreement between our analytical expressions for the CRLBs and their numerical values computed in [2] for QPSK, 16-QAM and 64-QAM signals. Furthermore, we see that the MCRB is essentially tight to the true CRLB at high SNR values as their ratio is close to 1. Therefore, due to the ease of derivation of the analytical MCRLBs, it is interesting to use these bounds as a benchmark to evaluate the performance of unbiased estimators. Whereas, it can be seen that at low SNR values, the MCRLB is a looser bound compared to the true CRLB. sequence is known, which is smaller than the true CRB pertaining to the estimate of when the data sequence is unknown. On the other hand, as shown in Figs (1), (2) and (3), we see in this SNR region, over which the lower bounds are almost the same, that the CRLB differs widely from one

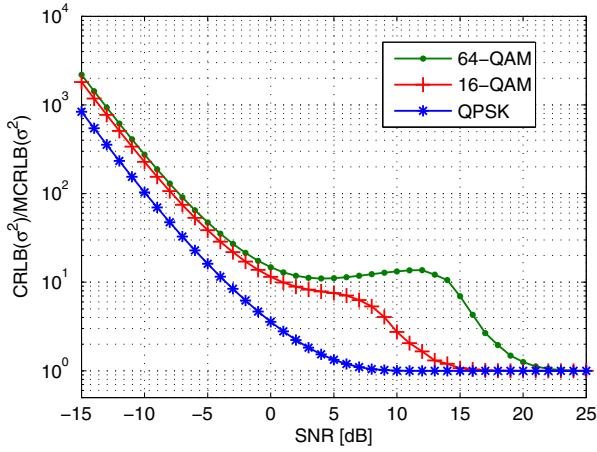


Fig. 3. CRLB(σ^2)/MCRB(σ^2) at different SNR values.

modulation order to another. In fact, this is directly related to the difference in power efficiency of each modulation.

V. CONCLUSIONS

In this paper, we derived closed-form expressions for the CRLBs and the MCRBs of the NDA parameter estimates of arbitrary square QAM signals. These lower bounds serve as benchmarks for the achievable performances of actual estimators. The frequency and phase CRLBs computed using our analytical expressions are similar to those computed empirically or numerically in other previous works. Moreover, Our proposed analytical expressions were compared coincide with those previously derived in the particular cases of quaternary phase-shift keying (QPSK) [3] only. Furthermore, we showed that the MCRBs and the CRLBs coincide for high SNR values. Therefore, computing the MCRB in this SNR region is more interesting since it is simpler to evaluate. Finally, the newly derived expressions are of great value in that they allow to quantify and analyze the achievable performance on the carrier phase, the carrier frequency, the noise power and the signal amplitude estimation in rectangular square QAM transmission.

APPENDIX PROOF OF (16)

In this appendix, we will the proof of $E \left\{ \frac{\partial^2 \ln(P[\mathbf{y}; \boldsymbol{\theta}])}{\partial \nu \partial S} \right\} = 0$ and equivalently, we prove (16). In fact, In fact, it can be seen from (15) that averaging $\frac{\partial^2 \ln(P[\mathbf{y}; \boldsymbol{\theta}])}{\partial S \partial \nu}$ with respect to $P[\mathbf{y}; \boldsymbol{\theta}]$ yields the following result :

$$E \left\{ \frac{\partial^2 \ln(P[\mathbf{y}; \boldsymbol{\theta}])}{\partial S \partial \nu} \right\} = \sum_{k=k_0}^{k_0+K-1} E \left\{ \frac{\partial^2 \ln(F_{\theta}(U(k)))}{\partial S \partial \nu} \right\} + \sum_{k=k_0}^{k_0+K-1} E \left\{ \frac{\partial^2 \ln(F_{\theta}(V(k)))}{\partial S \partial \nu} \right\}, \quad (50)$$

Moreover, we have :

$$\frac{\partial F_{\theta}(U(k))}{\partial \nu} = -V(k)G_{\theta}(U(k)), \quad (51)$$

$$\frac{\partial F_{\theta}(V(k))}{\partial \nu} = U(k)G_{\theta}(V(k)), \quad (52)$$

where

$$G_{\theta}(U(k)) = 2\pi k \sum_{i=1}^{2^p-1} \frac{2S(2i-1)d_p}{\sigma^2} e^{-\frac{S^2(2i-1)^2 d_p^2}{\sigma^2}} \sinh \left(\frac{(2i-1)d_p 2SU(k)}{\sigma^2} \right). \quad (53)$$

Thus, we obtain

$$\frac{\partial^2 F_{\theta}(U(k))}{\partial \nu \partial S} = -V(k) \frac{\partial G_{\theta}(U(k))}{\partial S}, \quad (54)$$

$$\frac{\partial^2 F_{\theta}(V(k))}{\partial \nu \partial S} = U(k) \frac{\partial G_{\theta}(V(k))}{\partial S}. \quad (55)$$

Injecting (54) and (55) into (50), we obtain the following result :

$$E \left\{ \frac{\partial^2 \ln(P[\mathbf{y}; \boldsymbol{\theta}])}{\partial S \partial \nu} \right\} = \sum_{k=k_0}^{k_0+K-1} E \left\{ \frac{\partial}{\partial S} \left(\frac{G_{\theta}(V(k))}{F_{\theta}(V(k))} \right) \right\} - \sum_{k=k_0}^{k_0+K-1} E \left\{ \frac{\partial}{\partial S} \left(\frac{G_{\theta}(U(k))}{F_{\theta}(U(k))} \right) \right\} \quad (56)$$

On the other hand, we show that $U(k)$ and $V(k)$ are two independent random variables which are identically distributed. Hence, we have :

$$E \left\{ \frac{\partial^2 \ln(p[\mathbf{y}; \boldsymbol{\theta}])}{\partial S \partial \nu} \right\} = 0. \quad (57)$$

Equivalently, we show in (16) that $E \left\{ \frac{\partial^2 \ln(P[\mathbf{y}; \boldsymbol{\theta}])}{\partial \theta_i \partial \theta_i} \right\}$ is equal to zero for the decoupled parameters $[\nu \ \phi]^T$ and $[S \ \sigma^2]^T$.

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