

# Cramér-Rao Bound for NDA DOA Estimates of Square QAM-Modulated Signals

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**Abstract**—This paper addresses the stochastic Cramér-Rao lower bound (CRLB) for the non-data-aided (NDA) direction of arrival (DOA) estimation of square quadrature amplitude (QAM)-modulated signals when the transmitted symbols are supposed to be completely unknown to the receiver. These signals are assumed to be corrupted by additive white circular complex Gaussian noise (AWCCGN). The channel is supposed to be slowly time-varying so that it can be assumed constant over the observation interval. The main contribution of this paper consists in deriving an explicit expression for the Fisher information matrix (FIM) in the case of a single square QAM-modulated waveform and an analytical expression for the stochastic CRLB of the NDA DOA estimates. It will be shown that the achievable performance on the DOA estimates hold almost the same irrespectively of the modulation order<sup>1</sup>.

## I. INTRODUCTION

Direction of arrival (DOA) estimation for multiple plane waves impinging on an arbitrary array of sensors has attracted much attention in the literature [1,2] due to its numerous applications in radar, sonar and communication. In particular, many NDA estimators of the DOA of a single signal waveform have been extensively studied. The performance of any DOA estimator is often assessed by computing and plotting its bias and variance as a function of the true SNR values. A given estimator is usually said to outperform another one, over a given SNR range, if it is unbiased and if it has a lower variance.

In signal processing, a well-known common lower bound for the variance of the unbiased estimators is the CRLB. It serves as a useful benchmark for practical estimators [3]. The CRLB is often numerically or empirically computed but even when a closed-form expression can be obtained, it is usually complex and requires tedious algebraic manipulations. Because the derivation of the stochastic CRLB was thought to be prohibitive in [4, (2.13)], the authors considered arbitrary deterministic signals corrupted by circular complex Gaussian noise. Hence, the likelihood function of the received samples remains Gaussian and the associated deterministic CRLB was easily derived. But, contrarily to the deterministic CRLB which is known to be not achievable in the general case, the stochastic CRLB can be achieved asymptotically (in the number of measurements) by several high resolution methods, such as the stochastic maximum likelihood (ML) estimator.

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Most of the contributions on the stochastic CRLB are dedicated to circular Gaussian distributed signals [5, 6] for which it can be easily derived, then extended to circular complex Gaussian distributions [7] which are often encountered in many applications. However, noncircular complex signals are also frequently encountered in digital communications and the corresponding CRLBs are therefore required in practice. But, to the best of our knowledge, no contributions have dealt with the stochastic CRLB of DOA estimates with higher-order square QAM constellations yet, except some recent results which are only applicable for binary phase-shift keying (BPSK) and quaternary phase-shift keying (QPSK) signals corrupted by additive circular complex Gaussian noise [8].

In this paper, we derive explicit expressions for the stochastic inphase/quadrature (I/Q) CRLB of the NDA DOA estimates with arbitrary square QAM modulated-signals. These signals are assumed to be AWCCGN-corrupted. The final results introduced in this paper generalize the elegant CRLB expressions of the NDA DOA estimates for BPSK- and QPSK-modulated signals, presented in [8], to higher-order rectangular square QAM modulations.

This paper is organized as follows. In section II, we introduce the system model that will be used throughout the article. In section III, an explicit expression for the FIM of the DOA parameter will be given and a simple explicit expression for the stochastic I/Q CRLB of the DOA estimates will be derived. In section IV, some concluding remarks will be drawn out. Throughout this paper, matrices and vectors are represented by bold upper case and bold lower case characters, respectively.

## II. SYSTEM MODEL

Consider a square QAM-modulated signal impinging on an arbitrary array of  $M$  sensors. Over the observation interval, the channel is supposed to be of a constant gain coefficient  $S$  and assumed to introduce a constant distortion phase  $\phi$ . We assume that the received signal is AWCCGN-corrupted where the noise power is  $\sigma^2$ . Assuming an ideal receiver with Nyquist shaping, ideal sample timing and perfect synchronization, the received signal at the output of the array matched filter can be modelled as a complex signal as follows :

$$\mathbf{y}(n) = S e^{j\phi} x(n) \mathbf{a} + \mathbf{w}(n), n = 1, 2, \dots, N, \quad (1)$$

where, at time index  $n$ ,  $x(n)$  is the transmitted symbol and  $\mathbf{a}$  is the steering vector parametrized by the scalar DOA para-

meter  $\theta$ ,  $\mathbf{a} = [1, e^{j\theta}, e^{2j \sin(\theta)}, \dots, e^{j(M-1) \sin(\theta)}]^T$ . We have  $\|\mathbf{a}\|^2 = M$ , where  $\|\cdot\|$  returns the second norm of any vector. The transmitted symbols  $\{x(n)\}_{n=1,2,\dots,N}$  are assumed to be independent from the noise components  $\{\mathbf{w}(n)\}_{n=1,2,\dots,N}$ . The noise components  $\{\mathbf{w}(n)\}_{n=1,2,\dots,N}$  are independent and identically distributed  $M$ -variate zero-mean complex circular Gaussian random vectors with independent real and imaginary parts and  $E\{\mathbf{w}(n)\mathbf{w}(n)^H\} = \sigma^2 \mathbf{I}_M$ .  $N$  represents the number of the received samples in the observation window. Moreover, the transmitted symbols are assumed to be independent and identically distributed (i.i.d) and drawn from any  $L$ -ary rectangular square QAM constellation, i.e.,  $L = 2^{2p}$  ( $p = 1, 2, 3 \dots$ ). To derive standard CRLBs, the squared QAM constellation energy is supposed to be normalized to one, i.e.,  $E\{|x(n)|^2\} = 1$ , where  $E\{\cdot\}$  refers to the expectation of any random variable and  $|\cdot|$  returns the module of any complex number.

In this work, all the parameters  $\sigma, S, \phi$  and  $\theta$  are considered to be unknown but deterministic. Therefore, we define the following parameter vector :

$$\boldsymbol{\alpha} = [\sigma \quad S \quad \phi \quad \theta]^T. \quad (2)$$

where the superscript  $T$  stands for the transpose operator. Moreover, we define the true SNR as follows :

$$\rho = \frac{S^2}{\sigma^2}. \quad (3)$$

### III. DERIVATION OF THE STOCHASTIC I/Q CRLB FOR SQUARE QAM-MODULATED SIGNALS

In this section, we will develop the stochastic FIM associated with the four considered parameters ( $\sigma, S, \phi, \theta$ ) of the square QAM-modulated and AWCCGN-corrupted signals. Then, we will deduce the analytical expressions for the stochastic CRLB of the NDA DOA estimates. We show (see appendix) that this FIM is given by the following block diagonal matrix :

$$\mathbf{I}_F(\boldsymbol{\alpha}) = \begin{pmatrix} \mathbf{I}_{F_1}(\sigma, S) & 0 \\ 0 & \mathbf{I}_{F_2}(\phi, \theta) \end{pmatrix}, \quad (4)$$

where, since  $\{\mathbf{y}(n)\}_{n=1,2,\dots,N}$  are i.i.d  $M$ -dimensional random variables, the elements of the matrix  $\mathbf{I}_{F_1}(\sigma, S)$  and  $\mathbf{I}_{F_2}(\phi, \theta)$  are given, respectively, by (5) and (6) :

$$[\mathbf{I}_{F_1}]_{i,l} = -N E \left\{ \frac{\partial^2 \ln(P[\mathbf{y}(n); \boldsymbol{\alpha}])}{\partial \alpha_i \partial \alpha_l} \right\}, \quad i, l = 1, 2, \quad (5)$$

$$[\mathbf{I}_{F_2}]_{i,l} = -N E \left\{ \frac{\partial^2 \ln(P[\mathbf{y}(n); \boldsymbol{\alpha}])}{\partial \alpha_{i+2} \partial \alpha_{l+2}} \right\}, \quad i, l = 1, 2. \quad (6)$$

The expectation  $E\{\cdot\}$  is taken with respect to  $\mathbf{y}(n)$  and  $\{\alpha_i\}_{i=1,2,3,4}$  are the elements of the parameter vector  $\boldsymbol{\alpha}$  given by (2).

Actually, the analytical derivations of the 8 elements involved in (5) and (6) require tedious algebraic manipulations. We now give the major final results and the algebraic developments required for the averaging step will be detailed

in the appendix. In fact, under the assumptions made so far and for any  $L$ -ary QAM constellation, it can be shown that the probability  $P[\mathbf{y}(n); \boldsymbol{\alpha}]$  of the received vector  $\mathbf{y}(n)$  parameterized by the parameter vector  $\boldsymbol{\alpha}$  is given by :

$$P[\mathbf{y}(n); \boldsymbol{\alpha}] = \frac{1}{L\pi^M \sigma^{2M}} \sum_{l=1}^L e^{-\|\mathbf{y}(n) - S e^{j\phi} x(n) \mathbf{a}\|^2 / \sigma^2}. \quad (7)$$

Moreover, it can be shown that this probability can also be written as :

$$P[\mathbf{y}(n); \boldsymbol{\alpha}] = \frac{1}{L\pi^M \sigma^{2M}} \exp \left\{ -\frac{\|\mathbf{y}(n)\|^2}{\sigma^2} \right\} D_{\boldsymbol{\alpha}}(n), \quad (8)$$

where  $D_{\boldsymbol{\alpha}}(n)$  is given by :

$$D_{\boldsymbol{\alpha}}(n) = \sum_{c_k \in \mathcal{C}} e^{-\frac{S^2 M |c_k|^2}{\sigma^2}} \exp \left\{ \frac{S \Re\{\mathbf{y}(n)^H e^{j\phi} c_k \mathbf{a}\}}{\sigma^2} \right\}, \quad (9)$$

where  $\mathcal{C}$  is the constellation alphabet and  $\Re\{\cdot\}$  returns the real part of any complex number.

In this paper, considering only square QAM constellations, i.e.,  $L = 2^{2p}$  ( $p = 1, 2, 3 \dots$ ), we derive analytical expressions for the CRLBs as a function of the true SNR values  $\rho$ . In fact, the major advantage offered by the special case of these square constellations is that  $D_{\boldsymbol{\alpha}}(n)$  and therefore  $P[\mathbf{y}(n); \boldsymbol{\alpha}]$  can be factorized, making it possible to obtain closed-form expressions, as a function of the true SNR  $\rho$ , for the FIM elements given by (4). Indeed, when  $L = 2^{2p}$  for any  $p \geq 1$ , we have  $\mathcal{C} = \{\pm(2i-1)d_p \pm j(2k-1)d_p\}_{i,k=1,2,\dots,2^{p-1}}$  where  $j^2 = -1$  and  $2d_p$  is the intersymbol distance in the I/Q plane. Therefore, after some algebraic manipulations, we show that  $D_{\boldsymbol{\alpha}}$  can be written as

$$D_{\boldsymbol{\alpha}}(n) = 4F_{\boldsymbol{\alpha}}(g(\mathbf{y}(n)))F_{\boldsymbol{\alpha}}(k(\mathbf{y}(n))), \quad (10)$$

where

$$g(\mathbf{y}(n)) = 2 \Re\{e^{j\phi} \mathbf{y}(n)^H \mathbf{a}\}, \quad (11)$$

$$k(\mathbf{y}(n)) = 2 \Im\{e^{j\phi} \mathbf{y}(n)^H \mathbf{a}\}, \quad (12)$$

$$F_{\boldsymbol{\alpha}}(t) = \sum_{i=1}^{2^{p-1}} \exp \left\{ -\frac{S^2 M (2i-1)^2 d_p^2}{\sigma^2} \right\} \cosh \left( \frac{(2i-1)d_p S t}{\sigma^2} \right), \quad (13)$$

where  $\Im\{\cdot\}$  refers to the imaginary part of any complex number. Moreover, for a normalized square rectangular QAM constellation,  $d_p$  is given by [9] :

$$d_p = \frac{2^{p-1}}{\sqrt{2^p \sum_{k=1}^{2^{p-1}} (2k-1)^2}}. \quad (14)$$

Consequently,  $P[\mathbf{y}(n); \boldsymbol{\alpha}]$  is given by :

$$P[\mathbf{y}(n); \boldsymbol{\alpha}] = \frac{4 e^{-\|\mathbf{y}(n)\|^2 / \sigma^2}}{L\pi^M \sigma^{2M}} F_{\boldsymbol{\alpha}}(g(\mathbf{y}(n)))F_{\boldsymbol{\alpha}}(k(\mathbf{y}(n))), \quad (15)$$

and the log-likelihood function of the received samples reduces simply to :

$$\begin{aligned} \ln(P[\mathbf{y}_n; \boldsymbol{\alpha}]) &= \ln \left( \frac{4}{L\pi^M} \right) - 2M \ln(\sigma) \\ &+ \ln(F_{\boldsymbol{\alpha}}(g(\mathbf{y}(n)))) + \ln(F_{\boldsymbol{\alpha}}(k(\mathbf{y}(n)))) . \end{aligned} \quad (16)$$

Differentiating (16) according to (5) and (6) and tedious developments (see the appendix for more details) yield the following results :

$$\mathbf{I}_{F_1}(\sigma, S) = \frac{2NM}{2^{p-2}\sigma^2} \begin{pmatrix} I_{1,1} & \frac{2S}{\sigma} [2^{p-2} - H(\rho)] \\ \frac{2S}{\sigma} [2^{p-2} - H(\rho)] & F(\rho) \end{pmatrix} \quad (17)$$

$$\mathbf{I}_{F_2}(\phi, \theta) = \frac{2N\rho}{2^{p-2}} \begin{pmatrix} I_{3,3} & I_{3,4} \\ I_{3,4} & I_{4,4} \end{pmatrix}, \quad (18)$$

where

$$I_{1,1} = 2^{p-1} - 4\rho [A_2 d_p^2 - G(\rho)] - 2\rho^2 M d_p^4 M A_4, \quad (19)$$

$$I_{3,3} = -2^{p-2} M^2 \rho + M d_p^2 (1 + M\rho) Q(\rho), \quad (20)$$

$$I_{3,4} = -j \dot{\mathbf{a}}^H \mathbf{a} [2^{p-2} M\rho - d_p^2 (1 + M\rho) Q(\rho)], \quad (21)$$

$$I_{4,4} = -2^{p-2} \rho |\mathbf{a}^H \dot{\mathbf{a}}|^2 + d_p^2 (|\dot{\mathbf{a}}|^2 + \rho |\mathbf{a}^H \dot{\mathbf{a}}|^2) Q(\rho). \quad (22)$$

In (17)-(22),  $\{A_m\}_{m=2,4}$ ,  $F(\rho)$ ,  $G(\rho)$ ,  $H(\rho)$  and  $Q(\rho)$  are given by :

$$\dot{\mathbf{a}} = \frac{\partial \mathbf{a}}{\partial \theta}, \quad (23)$$

$$A_m = \sum_{k=1}^{2^{p-1}} (2k-1)^m, \quad m = 2, 4, \quad (24)$$

$$F(\rho) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{f_\rho^2(t)}{h_\rho(t)} e^{-\frac{t^2}{2}} dt, \quad (25)$$

$$G(\rho) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{g_\rho^2(t)}{h_\rho(t)} e^{-\frac{t^2}{2}} dt, \quad (26)$$

$$H(\rho) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{f_\rho(t) g_\rho(t)}{h_\rho(t)} e^{-\frac{t^2}{2}} dt, \quad (27)$$

$$Q(\rho) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{q_\rho^2(t)}{h_\rho(t)} e^{-\frac{t^2}{2}} dt, \quad (28)$$

where

$$f_\rho(t) = \sum_{k=1}^{2^{p-1}} e^{-(2k-1)^2 d_p^2 M\rho} \left[ (2k-1) d_p t \sinh \left( (2k-1) d_p \sqrt{2M\rho} t \right) - (2k-1)^2 d_p^2 \sqrt{2M\rho} \cosh \left( (2k-1) d_p \sqrt{2M\rho} t \right) \right], \quad (29)$$

$$g_\rho(t) = \sum_{k=1}^{2^{p-1}} e^{-(2k-1)^2 d_p^2 M\rho} \left[ (2k-1) d_p t \sinh \left( (2k-1) d_p \sqrt{2M\rho} t \right) - (2k-1)^2 d_p^2 \sqrt{\frac{M\rho}{2}} \cosh \left( (2k-1) d_p \sqrt{2M\rho} t \right) \right], \quad (30)$$

$$q_\rho(t) = \sum_{k=1}^{2^{p-1}} e^{-(2k-1)^2 d_p^2 M\rho} (2k-1) \sinh \left( (2k-1) d_p \sqrt{2M\rho} t \right), \quad (31)$$

$$h_\rho(t) = \sum_{k=1}^{2^{p-1}} e^{-(2k-1)^2 d_p^2 M\rho} \cosh \left( (2k-1) d_p \sqrt{2M\rho} t \right) \quad (32)$$

Then, since the FIM is block diagonal, the expression for the DOA CRLB is derived as :

$$\text{CRLB}(\theta) = \frac{2^{p-2}}{2N\rho \det(\mathbf{I}_{F_2})} I_{3,3}. \quad (33)$$

After some algebraic manipulations and using the identity  $\dot{\mathbf{a}}^H \mathbf{a} = -\mathbf{a}^H \dot{\mathbf{a}}$ , we obtain the explicit expression for the CRLB :

$$\begin{aligned} \text{CRLB}(\theta) &= \frac{2^{p-2}}{N\rho\gamma d_p^2 Q(\rho)}, \\ &= \frac{1}{N\gamma\rho} \frac{A_2}{Q(\rho)}, \end{aligned} \quad (34)$$

where  $\gamma$  is the purely geometrical factor  $2\dot{\mathbf{a}}^H \mathbf{\Pi}_\mathbf{a}^\perp \dot{\mathbf{a}}$  with  $\mathbf{\Pi}_\mathbf{a}^\perp = \mathbf{I}_M - \mathbf{a}\mathbf{a}^H/M$  as defined in [8]. In the absence of the phase offset or assuming that it was perfectly recovered (i.e., the parameter  $\phi$  is supposed to be known), the CRLB for the DOA parameter becomes <sup>2</sup> :

$$\text{CRLB}^{CO}(\theta) = \frac{2^{p-2}}{2N\rho\Psi_\rho(\theta)}, \quad (35)$$

where

$$\Psi_\rho(\theta) = -2^{p-2} \rho |\mathbf{a}^H \dot{\mathbf{a}}|^2 + d_p^2 (|\dot{\mathbf{a}}|^2 + \rho |\mathbf{a}^H \dot{\mathbf{a}}|^2) Q(\rho). \quad (36)$$

By replacing  $p$  by 1 in (34) and (35), we obtain the same expressions presented in [8] in the special case of 4-QAM constellation. Moreover, we notice in (34) and (35) that the DOA CRLBs are inversely proportional to  $N$ . Therefore, they can be easily deduced for any observation interval size  $N_2$  if they were already computed for a given window size  $N_1$ , by scaling by the factor  $\frac{N_1}{N_2}$ .

#### IV. GRAPHICAL REPRESENTATIONS

In this section, we include some graphical representations of the lower bounds as a function of the SNR given by (34) for different modulation orders, different values of the number of sensors  $M$  and for a fixed DOA parameter  $\theta = 0$ . The receiving antennas are assumed to be distributed as a ULA.

In Fig. 1, we consider the same expression for the steering vector adopted in [8] :

$$\mathbf{a} = [1, e^{j\theta}, e^{2j\theta}, \dots, e^{j(M-1)\theta}]^T,$$

just for comparison purposes. It can be verified there that for  $p = 1$  there is a perfect agreement between our analytical expressions and those derived in the special case of QPSK by Delmas and Abeida in [8]. These graphical representations will be compared to some other graphical representations of

2. The superscript CO in  $\text{CRLB}^{CO}(\theta)$  means coherent.

the CRLB in the absence of the phase offset (35). In fact, for  $L = 4$ -,  $16$ -,  $64$ - and  $256$ -QAM, Figs. 1, 2, 3 and 4 show the corresponding CRLB and  $\text{CRLB}^{CO}$  as a function of the true SNR values for two different values of the number of sensors  $M = 3$  and  $M = 8$ .

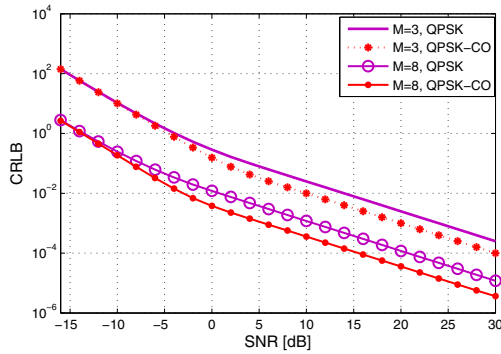


Fig. 1. CRLBs and  $\text{CRLB}^{CO}$  for the DOA estimates, 4-QAM,  $N = 1$ .

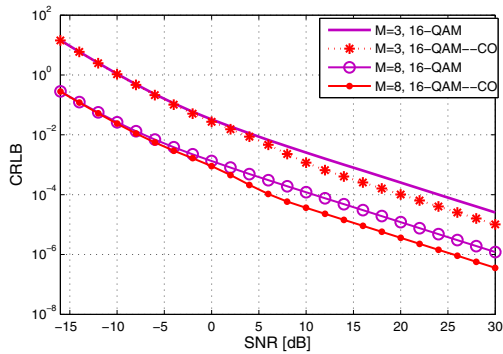


Fig. 2. CRLBs and  $\text{CRLB}^{CO}$  for the DOA estimates, 16-QAM,  $N = 1$ .

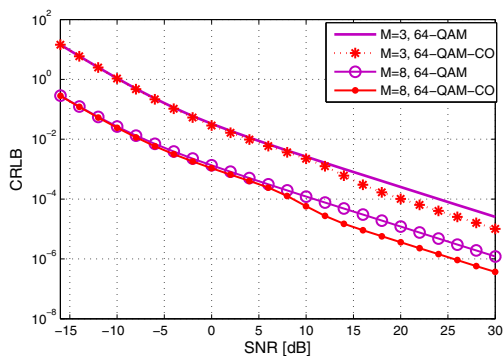


Fig. 3. CRLBs and  $\text{CRLB}^{CO}$  for the DOA estimates, 64-QAM,  $N = 1$ .

We see from Figs. 1, 2, 3 and 4 that, for relatively high SNR values, the CRLBs obtained in the absence of a phase offset are lower than those obtained when all the parameters

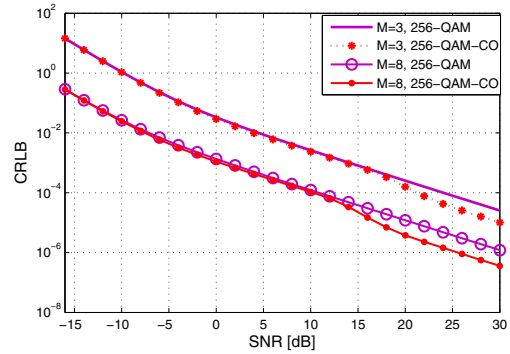


Fig. 4. CRLBs and  $\text{CRLB}^{CO}$  for the DOA estimates, 256-QAM,  $N = 1$ .

are assumed to be unknown. This is hardly surprising since the more information we exploit, the lower is the bound. The knowledge of the phase offset is obviously more informative about the unknown DOA parameter for high SNR values. Moreover, we see clearly that the difference between these two CRLBs decreases as the modulation order  $p$  increases.

We also see that the CRLBs decrease when the number of sensors  $M$  increases. In fact, the more sensors we have, the more samples we receive and consequently the more information we exploit. Finally, we notice from these figures that the achievable performance on the DOA estimates hold almost the same irrespectively of the modulation order. This can be analytically explained by examining the explicit expressions of the considered CRLBs in (34). Indeed, the modulation order  $p$  (or  $L = 2^{2p}$ ) appears only in the ratio  $\frac{A_2}{Q(\rho)}$  which turns out as shown in Fig. 5 to be the same for all the square QAM constellations.

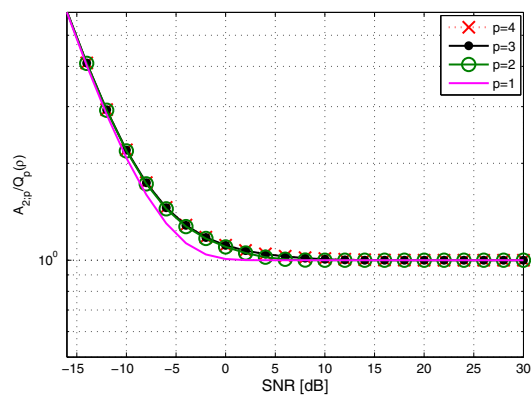


Fig. 5. Ratio  $A_2/Q(\rho)$  for  $p = 1, 2, 3$  and  $4$ .

## V. CONCLUSION

In this paper, we developed analytical expressions for the stochastic CRLBs for the NDA DOA estimates of rectangular square QAM-modulated signals corrupted by additive white circular complex Gaussian noise as a function of the true

SNR values. These stochastic CRLBs were compared with the CRLBs obtained in the absence of any phase offset introduced by the channel. Finally, we proved that the achievable performance on the DOA estimates hold almost the same irrespectively of the modulation order.

#### APPENDIX

As it was mentioned before, the derivation of the CRLB involves tedious algebraic manipulations. These mainly consist in the derivation of the Fisher information matrix elements. In this section, we will detail the main required algebraic manipulations to compute the expectation of the second derivatives which parallel in part some manipulations introduced in [8] for the special case of QPSK signals. In fact, as it can be seen from (16), due to the factorization of the received samples probability (10), the log-likelihood function involves the sum of two analogous terms. This reduces the complexity of the derivation of the second partial derivatives and their expected values. Indeed, we demonstrate that for  $i, l = 1, 2, 3$  and 4 :

$$\mathbb{E} \left\{ \frac{\partial^2 \ln(F_{\alpha}(g(\mathbf{y}(n))))}{\partial \alpha_i \partial \alpha_l} \right\} = \mathbb{E} \left\{ \frac{\partial^2 \ln(F_{\alpha}(k(\mathbf{y}(n))))}{\partial \alpha_i \partial \alpha_l} \right\}. \quad (37)$$

To derive an explicit expression for  $\mathbb{E} \left\{ \frac{\partial^2 \ln(F_{\alpha}(g(\mathbf{y}(n))))}{\partial \alpha_i \partial \alpha_l} \right\}$ , we average, as given by (38), with respect to the random variable  $z(n) = e^{j\phi} \mathbf{y}(n)^H \mathbf{a}$  whose probability  $P[z(n); \alpha] = P[(g(\mathbf{y}(n)), k_1(\mathbf{y}(n)); \alpha)]$  is given by (39) at the top of the next page. Then, using the regularity condition

$$\frac{\partial}{\partial \alpha_k} \int P[\mathbf{y}(n); \alpha] d\mathbf{y}(n) = \int \frac{\partial P[\mathbf{y}(n); \alpha]}{\partial \alpha_k} d\mathbf{y}(n) \quad (40)$$

which is fulfilled for any finite mixtures of Gaussian distributions, we have  $\mathbb{E} \left\{ \frac{\partial \ln(P[\mathbf{y}(n); \alpha])}{\partial \sigma} \right\} = 0$ . Furthermore, we have

$$\mathbb{E} \left\{ \frac{\partial \ln(P[\mathbf{y}(n); \alpha])}{\partial \sigma} \right\} = -\frac{2M}{\sigma} + \frac{2}{\sigma^3} \mathbb{E} \{ \|\mathbf{y}(n)\|^2 \} + 2 \mathbb{E} \left\{ \frac{\partial \ln(F_{\alpha}(g(\mathbf{y}(n))))}{\partial \sigma} \right\}. \quad (41)$$

Thus, we obtain :

$$\mathbb{E} \{ \|\mathbf{y}(n)\|^2 \} = M \sigma^2 - \sigma^3 \mathbb{E} \left\{ \frac{\partial \ln(F_{\alpha}(g(\mathbf{y}(n))))}{\partial \sigma} \right\}. \quad (42)$$

This identity enables us to derive the term  $(\mathbf{I}_{F_1})_{(\sigma, \sigma)}$ . To derive  $(\mathbf{I}_{F_2})_{(\phi, \phi)}$ , we show that

$$g(\mathbf{y}(n)) = 2SM \Re(x(n)) + e^{j\phi} \mathbf{w}(n)^H \mathbf{a} + e^{-j\phi} \mathbf{a}^H \mathbf{w}(n), \quad (43)$$

$$k(\mathbf{y}(n)) = \frac{2SM \Im(x(n)) + e^{j\phi} \mathbf{w}(n)^H \mathbf{a} - e^{-j\phi} \mathbf{a}^H \mathbf{w}(n)}{j}, \quad (44)$$

We take into account the hypothesis of the independence of  $\Re\{x(n)\}$  and  $\Im\{x(n)\}$ . We also have  $x(n)$  and the couple  $(e^{j\phi} \mathbf{w}(n)^H \mathbf{a} + e^{-j\phi} \mathbf{a}^H \mathbf{w}(n), \frac{1}{j}(e^{j\phi} \mathbf{w}(n)^H \mathbf{a} - e^{-j\phi} \mathbf{a}^H \mathbf{w}(n)))$  are independent and  $x(n)^*$  and  $(e^{j\phi} \mathbf{w}(n)^H \mathbf{a} + e^{-j\phi} \mathbf{a}^H \mathbf{w}(n), \frac{1}{j}(e^{j\phi} \mathbf{w}(n)^H \mathbf{a} - e^{-j\phi} \mathbf{a}^H \mathbf{w}(n)))$  are independent. Moreover, the two

terms of the couple  $(e^{j\phi} \mathbf{w}(n)^H \mathbf{a} + e^{-j\phi} \mathbf{a}^H \mathbf{w}(n)$  and  $\frac{1}{j}(e^{j\phi} \mathbf{w}(n)^H \mathbf{a} - e^{-j\phi} \mathbf{a}^H \mathbf{w}(n)))$  are uncorrelated Gaussian random variables and therefore independent. Hence,  $g(\mathbf{y}(n))$  and  $k(\mathbf{y}(n))$  are independent. This property allows us to calculate  $(\mathbf{I}_{F_2})_{(\phi, \phi)}$ .

To derive  $(\mathbf{I}_{F_2})_{(\theta, \theta)}$ , we consider the two terms  $\dot{g}(\mathbf{y}(n))$  and  $\ddot{g}(\mathbf{y}(n))$  with

$$\dot{g}(\mathbf{y}(n)) = \frac{\partial g(\mathbf{y}(n))}{\partial \theta} = 2\Re \{ e^{j\phi} \mathbf{y}(n)^H \dot{\mathbf{a}} \}, \quad (45)$$

$$\ddot{g}(\mathbf{y}(n)) = \frac{\partial^2 g(\mathbf{y}(n))}{\partial^2 \theta} = 2\Re \{ e^{j\phi} \mathbf{y}(n)^H \ddot{\mathbf{a}} \}. \quad (46)$$

We have :

$$\dot{g}(\mathbf{y}(n)) = S(x(n)^* \mathbf{a}^H \dot{\mathbf{a}} + x(n) \dot{\mathbf{a}}^H \mathbf{a}) + e^{j\phi} \mathbf{w}(n)^H \dot{\mathbf{a}} + e^{-j\phi} \dot{\mathbf{a}}^H \mathbf{w}(n), \quad (47)$$

$$\ddot{g}(\mathbf{y}(n)) = S(x(n)^* \mathbf{a}^H \ddot{\mathbf{a}} + x(n) \ddot{\mathbf{a}}^H \mathbf{a}) + e^{j\phi} \mathbf{w}(n)^H \ddot{\mathbf{a}} + e^{-j\phi} \ddot{\mathbf{a}}^H \mathbf{w}(n). \quad (48)$$

Then, replacing  $x(n)^*$  with  $\frac{g(\mathbf{y}(n)) + jk(\mathbf{y}(n)) - e^{j\phi} \mathbf{w}(n)^H \mathbf{a}}{2MS}$ ,  $x(n)$  with  $\frac{g(\mathbf{y}(n)) - jk(\mathbf{y}(n)) - e^{-j\phi} \mathbf{w}(n)^H \mathbf{a}}{2MS}$  in (47) and using  $\dot{\mathbf{a}}^H \mathbf{a} + \mathbf{a}^H \dot{\mathbf{a}} = 0$  derived from  $\|\mathbf{a}\|^2 = M$ , we prove that  $g(\mathbf{y}(n))$  and  $\dot{g}(\mathbf{y}(n))$  are independent. After some algebraic manipulations and using some properties of the complex Gaussian distribution, we obtain :

$$\mathbb{E} \{ (\dot{g}(\mathbf{y}(n)))^2 \} = 2\|\dot{\mathbf{a}}\|^2 \sigma^2 + 2S^2 |\mathbf{a}^H \dot{\mathbf{a}}|^2. \quad (49)$$

In the same way, by injecting  $x(n)^*$  and  $x(n)$  in (48) and by using the identity  $\ddot{\mathbf{a}}^H \mathbf{a} + \mathbf{a}^H \ddot{\mathbf{a}} + 2\|\dot{\mathbf{a}}\|^2 = 0$  derived from  $\|\mathbf{a}\|^2 = M$ , we obtain :

$$\ddot{g}(\mathbf{y}(n)) = -\frac{\|\dot{\mathbf{a}}\|^2}{M} g(\mathbf{y}(n)) + \frac{j}{2M} (\mathbf{a}^H \ddot{\mathbf{a}} - \ddot{\mathbf{a}}^H \mathbf{a}) k(\mathbf{y}(n)) + z'(n), \quad (50)$$

with

$$z'(n) = e^{j\phi} \mathbf{w}(n)^H \ddot{\mathbf{a}} + e^{-j\phi} \ddot{\mathbf{a}}^H \mathbf{w}(n) - \frac{1}{M} (e^{j\phi} \mathbf{w}(n)^H \mathbf{a} \mathbf{a}^H \ddot{\mathbf{a}} + e^{-j\phi} \mathbf{a}^H \mathbf{w}(n) \ddot{\mathbf{a}}^H \mathbf{a}). \quad (51)$$

By denoting  $z(n) = e^{j\phi} \mathbf{w}(n)^H \mathbf{a} + e^{-j\phi} \mathbf{a}^H \mathbf{w}(n)$  and  $z''(n) = e^{j\phi} \mathbf{w}(n)^H \ddot{\mathbf{a}} + e^{-j\phi} \ddot{\mathbf{a}}^H \mathbf{w}(n)$ , we have  $z(n)$ ,  $z'(n)$  and  $z''(n)$  are zero-mean Gaussian distributed with

$$\mathbb{E} \{ z(n) z'(n) \} = \mathbb{E} \{ z(n) z''(n) \} - \Gamma. \quad (52)$$

where

$$\Gamma = \frac{\mathbb{E} \{ z(n) (e^{j\phi} \mathbf{w}(n)^H \mathbf{a} \mathbf{a}^H \ddot{\mathbf{a}} + e^{-j\phi} \mathbf{a}^H \mathbf{w}(n) \ddot{\mathbf{a}}^H \mathbf{a}) \}}{M}. \quad (53)$$

$$\begin{aligned} \mathbb{E}(z(n) z'(n)) &= -2\|\dot{\mathbf{a}}\|^2 \sigma^2 - \frac{1}{M} (-2M\|\dot{\mathbf{a}}\|^2 \sigma^2) \\ &= 0. \end{aligned} \quad (54)$$

Consequently,  $z(n)$  and  $z'(n)$  are independent random variables. Since  $\Re\{x(n)\}$  and  $z'(n)$  are also independent,

$$\mathbb{E} \left\{ \frac{\partial^2 \ln (F_{\alpha}(g(\mathbf{y}(n))))}{\partial \alpha_i \partial \alpha_j} \right\} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\partial^2 \ln (F_{\alpha}(g(\mathbf{y}(n))))}{\partial \alpha_i \partial \alpha_j} P[(g(\mathbf{y}(n)), k(\mathbf{y}(n)); \alpha)] dg(\mathbf{y}(n)) dk(\mathbf{y}(n)), \quad (38)$$

where

$$P[(g(\mathbf{y}(n)), k(\mathbf{y}(n)); \alpha)] = \frac{1}{LM\pi\sigma^2} \exp \left\{ -\frac{g^2(\mathbf{y}(n)) + k^2(\mathbf{y}(n))}{4M\sigma^2} \right\} F_{\alpha}(g(\mathbf{y}(n))) F_{\alpha}(k(\mathbf{y}(n))). \quad (39)$$

$g(\mathbf{y}(n))$  and  $z'(n)$  are independent. Using these properties of  $\dot{g}(\mathbf{y}(n))$  and  $\ddot{g}(\mathbf{y}(n))$  and some algebraic manipulations, we evaluate  $(\mathbf{I}_{F_2})_{(\phi, \phi)}$ .

The same approach is used to derive  $(\mathbf{I}_{F_2})_{(\theta, \phi)}$ . In fact,  $\dot{g}(\mathbf{y}(n))k(\mathbf{y}(n))$  and  $g(\mathbf{y}(n))$  are independent. We have :

$$\mathbb{E} \{ \dot{g}(\mathbf{y}(n))k(\mathbf{y}(n)) \} = -j\dot{\mathbf{a}}^H \mathbf{a} (2S^2 M + 2\sigma^2). \quad (55)$$

Moreover, we consider

$$\begin{aligned} \dot{k}(\mathbf{y}(n)) &= \frac{\partial k(\mathbf{y}(n))}{\partial \theta} = 2\Im \{ e^{j\phi} \mathbf{y}(n)^H \dot{\mathbf{a}} \}, \\ &= \frac{j\dot{\mathbf{a}}^H \mathbf{a}}{M} g(\mathbf{y}(n)) + r'(n), \end{aligned} \quad (56)$$

with

$$\begin{aligned} r'(n) &= \frac{j}{M} (e^{j\phi} \mathbf{w}(n)^H \mathbf{a} \mathbf{a}^H \dot{\mathbf{a}} - e^{-j\phi} \dot{\mathbf{a}}^H \mathbf{w}(n) \dot{\mathbf{a}}^H \mathbf{a}) - \\ & j(e^{j\phi} \mathbf{w}(n)^H \dot{\mathbf{a}} - e^{-j\phi} \dot{\mathbf{a}}^H \mathbf{w}(n)). \end{aligned} \quad (57)$$

$r'(n)$  is zero-mean Gaussian distributed with  $\mathbb{E}\{z(n)r'(n)\} = 0$ .

Finally, since  $k(\mathbf{y}(n))$  and  $\dot{g}(\mathbf{y}(n))$  are zero-mean,  $(\mathbf{I}_F)_{(S, \phi)} = (\mathbf{I}_F)_{(\sigma, \phi)} = (\mathbf{I}_F)_{(S, \theta)} = (\mathbf{I}_F)_{(\sigma, \theta)} = 0$ .

## REFERENCES

- [1] D. H. Johnson, and D. E. Dudgeon, *Array Signal Processing Concepts and Techniques*, Prentice Hall, Englewood Cliffs, NJ, 1993.
- [2] M. Wax, *Detection and Estimation of Superimposed Signals*, Ph.D. Dissertation, Stanford University, 1982.
- [3] P. Stoica, A.G. Larsson, and A.B. Gershman, "The Stochastic CRB for Array Processing : A textbook derivation," *IEEE Signal Process. Lett.*, vol. 8, pp. 148-150, May. 2001.
- [4] P. Stoica and A. Nehorai, "Performance Study of Conditional and Unconditional Direction of Arrival Estimation," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 38, pp. 1783-1795, Oct. 1990.
- [5] D. Slepian, "Estimation of Signal Parameters in the Presence of Noise," *Trans. IRE Prof. Group Inform. Theory PG IT-3*, pp. 68-89, 1954.
- [6] W. J. Bangs, *Array Processing with Generalized Beamformers*, Ph.D. Dissertation, Yale Univ., New Haven, CT, 1971.
- [7] P. Stoica and R. Moses, *Introduction to Spectral Analysis. Upper Saddle River, NJ : Prentice-Hall*, 1997.
- [8] J. P. Delmas and H. Abeida, "Cramér-Rao bounds of DOA estimates for BPSK and QPSK modulated signals," *IEEE Trans Sig Process*, vol. 54, no. 1, January. 2006.
- [9] F. Bellili, A. Stéphenne and S. Affes, "Cramér-Rao bound for NDA SNR estimates of square QAM modulated signals," in *Proc IEEE WCNC 2009, April 2009*.