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Cramér-Rao Bound for NDA SNR Estimates of Square QAM Modulated Signals

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Abstract— **In this paper, we derive analytical expressions for the inphase/quadrature Cramér-Rao lower bounds (I/Q CRLB) for the non-data-aided (NDA) signal-to-noise ratio (SNR) estimation of square quadrature amplitude modulated (QAM) signals. The channel is supposed to be slowly time-varying so that it can be assumed constant over the observation interval. The signal is assumed to be corrupted by additive white Gaussian noise (AWGN). We will demonstrate that the shape of the bounds differs greatly as the modulation order changes.**

I. Introduction

For many applications in modern communication systems, it is often a requirement to obtain an estimate of the SNR [1, 2]. When such an SNR estimate is obtained using only the received samples, without knowledge of the transmitted symbols, the SNR estimation process is referred to as non-data-aided (NDA). Therefore, numerous methods of NDA SNR estimation were introduced during the past few decades. Using Monte Carlo simulations, the performance of any SNR estimator is often assessed by computing and plotting its bias and variance as a function of the SNR true values. A given estimator is usually said to outperform another, over a given range of SNR, if it is unbiased and it has a lower variance.

A well-known common lower bound for the variance of any unbiased parameter estimator is the Cramér-Rao lower bound (CRLB). The CRLB is therefore very useful to statistically assess the performance of a given unbiased estimator in terms of its variance. The derivation of the CRLB is sometimes not tractable and it must therefore be computed numerically. But even when a closed-form expression can be obtained, it is usually complex and requires tedious algebraic manipulations.

The envelope-based CRLBs for NDA SNR estimates of QAM-modulated signals are derived in [3]. However, these theoretical limits of the envelope-based estimators do not use the whole information carried by the inphase and quadrature (I/Q) components of the received signal. Consequently, they differ from those of estimators which exploit the whole information carried in the inphase and quadrature components. The I/Q CRLB for data-aided (DA) SNR estimates of QAM signals, for which all the transmitted symbols are supposed to be perfectly known to the receiver, is derived in [4]. On the other hand, the I/Q CRLBs for NDA SNR estimates of BPSK and QPSK modulated signals, for which the transmitted symbols are supposed to be completely unknown to the receiver, were derived in [5].

In [6] and [7] recently published and accepted, respectively, the I/Q CRLBs for NDA SNR estimates of QAM-modulated signals were considered but numerically or empirically computed from very complex expressions. In this paper, considering square QAM constellations which are the most popular, and using the inphase and quadrature components of the received signal, we derive relatively simple analytical expressions for the CRLBs of NDA SNR estimates in AWGN channels. The final results introduced in this paper generalize the elegant CRLB expressions for the SNR estimates of BPSK and QPSK modulated signals presented in [5] to higher-order square QAM modulations.

The remainder of this paper will be organized as follows. In section II, we will introduce the system model that will be used throughout the article. In section III, we will derive the expressions for these I/Q CRLBs. Finally, before concluding this paper, we will present and comment some graphical representations of our analytical expressions in IV.

II. System Model

Consider a traditional digital communication system broadcasting and receiving a QAM modulated signal. The channel is supposed to be of a constant gain coefficient S over the observation interval. We assume that we receive an AWGN-corrupted signal where the noise power is $2\sigma^2$. Assuming an ideal receiver with perfect synchronization, the received signal at the output of the matched filter can be modelled as a complex signal as follows :

$$y(n) = S\ a(n)\ e^{j\phi} + w(n), n = 1, 2, \ldots, N, \qquad (1)$$

where, at time index n, $a(n)$ is the transmitted symbol and $y(n)$ is the corresponding received sample. The noise component $w(n)$ is modelled by a zero-mean Gaussian random variable with independent real and imaginary parts, each of variance σ^2. N is the number of received samples in the observation interval. Moreover, the transmitted symbols are assumed to be independent and identically distributed (iid) and drawn from an M-ary non-constant square QAM constellation, where $M = 2^{2p}$ $(p = 1, 2, 3 \ldots)$. ϕ accounts for any non-random phase shift introduced by the channel. In addition, to derive standard CRLBs, the squared QAM constellation energy is supposed to be normalized to one, i.e., $\mathrm{E}\{|a(n)|^2\} = 1$.

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Based on the N received samples, the true SNR that we wish to estimate is defined as

$$\rho \;=\; \frac{S^2}{2\sigma^2}. \tag{2}$$

From eq. (2), we see that there are two parameters which are involved in the derivation of the SNR CRLBs, S and σ^2. Hence, we define the following parameter vector :

$$\boldsymbol{\theta} = [S \quad \sigma^2]. \tag{3}$$

Moreover, since for clearer interpretation we usually use the decibel scale, we will henceforth consider the following parameter transformation :

$$g(\boldsymbol{\theta}) = 10\log_{10}\left(\frac{S^2}{2\sigma^2}\right). \tag{4}$$

III. Derivation of the I/Q CRLBs

In this section, we will derive analytical expressions for the NDA SNR estimates when the transmitted signal is QAM modulated and AWGN corrupted. As shown in [8], the CRLB for parameter transformations is given by

$$\text{CRLB}(\rho) = \frac{\partial g(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} I^{-1}(\boldsymbol{\theta}) \frac{\partial g(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}^T, \tag{5}$$

where the derivative of our parameter transformation $\partial g(\boldsymbol{\theta})/\partial\boldsymbol{\theta}$ is given by

$$\frac{\partial g(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \left[\frac{20}{\ln(10)S} \quad \frac{-10}{\ln(10)\sigma^2}\right], \tag{6}$$

and $I(\boldsymbol{\theta})$ is the Fisher information matrix (FIM) defined as

$$I(\boldsymbol{\theta}) = \begin{pmatrix} -\mathrm{E}\left\{\frac{\partial^2 \ln(P[\boldsymbol{y};\boldsymbol{\theta}])}{\partial S^2}\right\} & -\mathrm{E}\left\{\frac{\partial^2 \ln(P[\boldsymbol{y};\boldsymbol{\theta}])}{\partial S \partial \sigma^2}\right\} \\ -\mathrm{E}\left\{\frac{\partial^2 \ln(P[\boldsymbol{y};\boldsymbol{\theta}])}{\partial \sigma^2 \partial S}\right\} & -\mathrm{E}\left\{\frac{\partial^2 \ln(P[\boldsymbol{y};\boldsymbol{\theta}])}{\partial \sigma^2}\right\} \end{pmatrix}, \tag{7}$$

where $\boldsymbol{y}$ is a vector that contains the N received samples, i.e., $\boldsymbol{y} = [y(1), y(2), \cdots, y(N)]^T$, and $P[\boldsymbol{y}, \boldsymbol{\theta}]$ is the probability density function of $\boldsymbol{y}$ parameterized by $\boldsymbol{\theta}$. The expectation $\mathrm{E}\{.\}$ is taken with respect to $\boldsymbol{y}$.

Usually, the derivation of the CRLB involves tedious algebraic manipulations. These mainly consist in the derivation of the Fisher information matrix elements. We now give the major final results. In fact, under the assumptions made so far and for a general M-ary QAM constellation (i.e. $M = 2^q$ for arbitrary integer $q \geq 2$), it can be shown that the probability $P[y(n); \boldsymbol{\theta}]$ of the received sample $y(n)$ parameterized by $\boldsymbol{\theta}$ is given by :

$$P[y(n);\boldsymbol{\theta}] \;=\; \frac{1}{2M\pi\sigma^2}\exp\left\{-\frac{I(n)^2+Q(n)^2}{2\sigma^2}\right\} D_{\boldsymbol{\theta}}(n), \tag{8}$$

where $I(n)$ and $Q(n)$ are, respectively, the inphase/real and quadrature/imaginary component/part of the corresponding received sample $y(n)$, which means $y(n) = I(n) + jQ(n)$, and $D_{\boldsymbol{\theta}}(n)$ is given by :

$$D_{\boldsymbol{\theta}}(n) = \sum_{c_k\in\mathcal{C}} \exp\left\{-\frac{S^2||c_k||^2}{2\sigma^2}\right\} \exp\left\{\frac{S\Re\{y^*(n)e^{j\phi}c_k\}}{\sigma^2}\right\}, \tag{9}$$

where $\mathcal{C}$ is the constellation alphabet and $\Re\{.\}$ refers to the real part of any complex number.

In [6] and [7], no analytical expressions for the CRLBs as a function of the true SNR values ρ were provided. In fact, the FIM elements are, respectively, numerically and empirically computed then simply injected in (10) or (5) in order to evaluate these CRLBs. The major difficulty recognized by the authors of these two papers is the analytical derivation of the average of the second partial derivatives in (7) with respect to the received samples $y(n)$. Indeed, in [7], this averaging step was carried out using Monte Carlo simulations where samples of $y(n)$ were generated via computer simulation and the expected value computed empirically. However, in [6], where the parameters ρ and σ^2 are considered instead of S and σ^2 in our system model, the author considered the following CRLB expression :

$$\text{CRLB}(\rho) = \frac{\frac{1}{N}\,\mathrm{E}\left\{\left(\frac{\partial\Lambda}{\partial\sigma^2}\right)^2\right\}}{\mathrm{E}\left\{\left(\frac{\partial\Lambda}{\partial\rho}\right)^2\right\}\mathrm{E}\left\{\left(\frac{\partial\Lambda}{\partial\sigma^2}\right)^2\right\} - \left(\mathrm{E}\left\{\frac{\partial\Lambda}{\partial\rho}\frac{\partial\Lambda}{\partial\sigma^2}\right\}\right)^2}, \tag{10}$$

where $\Lambda(\boldsymbol{y};\rho,\sigma^2) = \ln(P[\boldsymbol{y};\rho,\sigma^2])$ is the log-likelihood function corresponding to the system model considered in [6]. But, before beeing able to evaluate the CRLBs, the expected values, with respect to $\boldsymbol{y}$, involved in (10), were numerically tackled using a Gauss-Hermitian quadrature.

However, in this paper, considering only square QAM constellations, we derive the analytical expressions for the CRLBs as a function of the true SNR values ρ. In fact, the major advantage offered by the special case of square QAM constellations is that $P[y(n), \boldsymbol{\theta}]$ can be factorized, making it possible to obtain simpler analytical expressions, as a function of ρ, for the FIM elements given by (7). Indeed, when $M = 2^{2p}$ for any p, we have $\mathcal{C} = \{\pm(2i-1)d_p \pm j(2k-1)d_p\}_{i,k=1,2,\cdots,2^{p-1}}$ where $j^2 = -1$. Therefore, we show that $D_{\boldsymbol{\theta}}$ can be written as

$$D_{\boldsymbol{\theta}}(n) \;=\; 4F_{\boldsymbol{\theta}}(U(n))F_{\boldsymbol{\theta}}(V(n)), \tag{11}$$

where

$$F_{\boldsymbol{\theta}}(t) = \sum_{i=1}^{2^{p-1}} \exp\left\{-\frac{S^2(2i-1)^2d_p^2}{2\sigma^2}\right\}\cosh\left(\frac{(2i-1)d_pSt}{\sigma^2}\right), \tag{12}$$

$$U(n) = I(n)\cos(\phi) + Q(n)\sin(\phi), \tag{13}$$

$$V(n) = I(n)\sin(\phi) - Q(n)\cos(\phi), \tag{14}$$

where $2d_p$ is the intersymbol distance in the I/Q plane. For a normalized square rectangular QAM constellation, d_p is computed using the following assumption :

$$\frac{\sum_{k=1}^{2^{2p}} ||c_k||^2}{2^{2p}} = 1, \quad (15)$$

which yields the following result :

$$d_p = \frac{2^{p-1}}{\sqrt{(2^p-1)\sum_{k=1}^{2^{p-1}}(2k-1)^2}}. \quad (16)$$

On the other hand, since the transmitted symbols are iid, then the corresponding AWGN-corrupted received samples are independent and the probability of the received vector $\boldsymbol{y} = [y(1), y(2), \cdots, y(N)]$ is given by :

$$P[\boldsymbol{y};\boldsymbol{\theta}] = \left(\frac{2}{M\pi\sigma^2}\right)^N \exp\left\{-\frac{\sum_{n=1}^N I(n)^2 + Q(n)^2}{2\sigma^2}\right\} \times \prod_{n=1}^N F_{\boldsymbol{\theta}}(U(n))F_{\boldsymbol{\theta}}(V(n)). \quad (17)$$

Finally, the log-likelihood function of the received samples is given by :

$$\ln(P[(\boldsymbol{y};\boldsymbol{\theta})]) = N\ln(2) - N\ln(M\pi\sigma^2) + \sum_{n=1}^N \ln\left(F_{\boldsymbol{\theta}}(U(n))\right) + \sum_{n=1}^N \ln\left(F_{\boldsymbol{\theta}}(V(n))\right). \quad (18)$$

As it can be seen from (18), due to the factorization of the received samples probability, as seen from (11), the log-likelihood function involves the sum of two analogous terms. This reduces the complexity of the derivation of the second partial derivatives and their expected values. In fact, differentiating (18) and tedious developments yield the following result :

$$I(\boldsymbol{\theta}) = \frac{N}{2^{p-2}\sigma^4}\begin{pmatrix} \sigma^2 F(\rho) & S[A_2 - H(\rho)] \\ S[A_2 - H(\rho)] & I_{2,2}\end{pmatrix}, \quad (19)$$

where

$$I_{2,2} = 2^{p-2} + \frac{S^2}{\sigma^2}[A_2 + G(\rho)] - \rho(4A_2 + \rho A_4). \quad (20)$$

In (19) and (20), A_2, A_4, $F(\rho)$, $G(\rho)$, and $H(\rho)$ are given by :

$$A_2 = \sum_{k=1}^{2^{p-1}}(2k-1)^2 d_p^2, \quad (21)$$

$$A_4 = \sum_{k=1}^{2^{p-1}}(2k-1)^4 d_p^4, \quad (22)$$

$$F(\rho) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{f_\rho^2(t)}{h_\rho(t)}\; e^{-\frac{t^2}{2}}dt, \quad (23)$$

$$G(\rho) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{g_\rho^2(t)}{h_\rho(t)}\; e^{-\frac{t^2}{2}}dt, \quad (24)$$

$$H(\rho) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{f_\rho(t)g_\rho(t)}{h_\rho(t)}\; e^{-\frac{t^2}{2}}dt, \quad (25)$$

where

$$f_\rho(t) = \sum_{k=1}^{2^{p-1}} e^{-(2k-1)^2 d_p^2\rho}\Big[(2k-1)d_p\;\, t\sinh\left((2k-1)d_p\sqrt{2\rho}\;\, t\right) - (2k-1)^2 d_p^2\sqrt{2\rho}\;\, t\cosh\left((2k-1)d_p\sqrt{2\rho}\;\, t\right)\Big], \quad (26)$$

$$g_\rho(t) = \sum_{k=1}^{2^{p-1}} e^{-(2k-1)^2 d_p^2\rho}\Big[(2k-1)d_p\;\, t\sinh\left((2k-1)d_p\sqrt{2\rho}\;\, t\right) - (2k-1)^2 d_p^2\sqrt{\frac{\rho}{2}}\;\, t\cosh\left((2k-1)d_p\sqrt{2\rho}\;\, t\right)\Big], \quad (27)$$

$$h_\rho(t) = \sum_{k=1}^{2^{p-1}} e^{-(2k-1)^2 d_p^2\rho}\cosh\left((2k-1)d_p\sqrt{2\rho}\;\, t\right). \quad (28)$$

Finally, injecting (6) and (19) in (5), we get the following result :

$$\text{CRB}_{\text{NDA}}(\rho) = \frac{2^{p-1}}{N\log^2(10)} \times \frac{A(\rho)}{B(\rho)}, \quad (29)$$

where

$$A(\rho) = 100\left[\frac{2^{p-2}}{\rho} + G(\rho) - 2A_2 - \rho A_4\right] - 200[H(\rho) - A_2] + 50F(\rho), \quad (30)$$

$$B(\rho) = F(\rho)\left[2^{p-2} + 2\rho G(\rho) - 2\rho A_2 - \rho^2 A_4\right] - 2\rho\left[A_2 - H(\rho)\right]^2. \quad (31)$$

As it can be seen from (29), for any square QAM constellation, the CRLBs do not depend on the phase ϕ introduced by the channel, as shown earlier in [5] for BPSK and QPSK modulated signals only. This general property was only explained intuitively in [6].

Finally, it is worth noting that the analytical expression for the CRLBs as a function of the true SNR, established in (29), generalizes to higher-order square QAM modulations the elegant CRLB expression derived in [5] for QPSK constellations where it was shown that

$$\text{CRLB}_{\text{QPSK}}(\rho) = \frac{100\left(\frac{2}{\rho} - f\left(\frac{\rho}{2}\right) + 1\right)}{N(\ln(10))^2\left(1 - f\left(\frac{\rho}{2}\right) - 2\rho f\left(\frac{\rho}{2}\right)\right)}, \quad (32)$$

where

$$f(\rho) = \frac{\exp(-\rho)}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{t^2}{\cosh(t\sqrt{2\rho})}e^{-\frac{t^2}{2}}dt. \quad (33)$$

IV. Graphical Representations

In this section, we include some graphical representations of the lower bounds given by (29) for different modulation orders. In fact, for $M = 4, 16, 64$ and 256, Figs. 1, 2, 3, and 4 show the CRLBs[dB^2] as a function of the true SNR values over the range [0dB, 20dB] for two different observation intervals of length $N = 100$ and $N = 1000$, respectively [1].

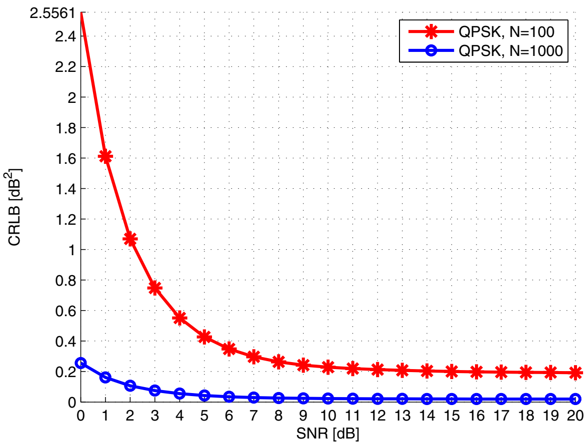

Fig. 1. CRLB of the SNR estimates, 4-QAM.

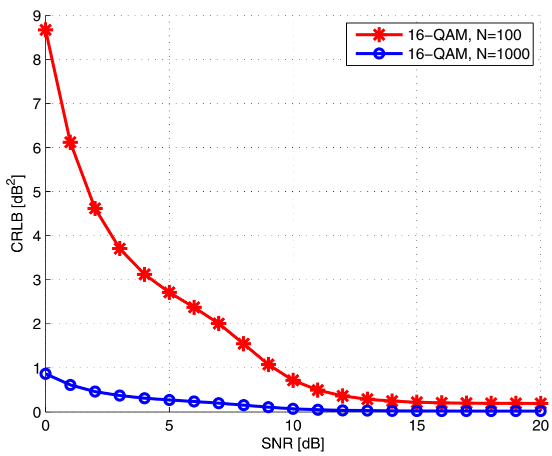

Fig. 2. CRLB of the SNR estimates, 16-QAM.

We see from these figures that the CRLBs decrease when the number of the received samples N increases. This is quite intuitive because the more information we exploit, the lower the bound. This is analytically verified as the final CRLBs expression is inversely proportional to the length of the observation interval. However, the CRLBs for $N = 100$ and $N = 1000$ are relatively lower and almost the same for high SNR values. This is because, in this SNR region, the useful signal is not too much corrupted by the additive noise and, consequently, the signal and noise powers can be estimated quite adequately, even with relatively few samples.

1. It should be noted that, in the special cases of 4-QAM and 16-QAM constellations, the CRLB[dB^2] curves plotted using our analytical expressions are, respectively, similar to those presented in [5] (QPSK) and provided by the empirical approach introduced in [7] (16-QAM).

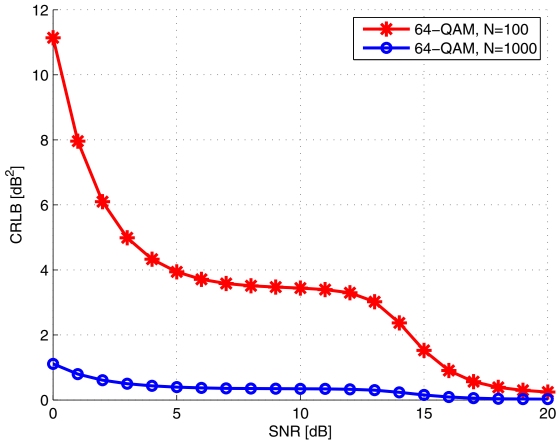

Fig. 3. CRLB of the SNR estimates, 64-QAM.

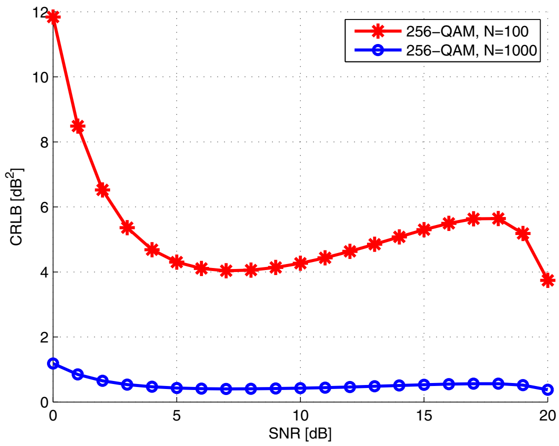

Fig. 4. CRLB of the SNR estimates, 256-QAM.

On the other hand, we see in this SNR region, over which the lower bounds are almost the same, that the CRLB differs widely from one modulation order to another. In fact, this is directly related to the difference in power efficiency of each modulation.

V. Conclusion

In this paper, we derived analytical expressions for the CRLBs of the NDA SNR estimates of square QAM-modulated signals as a function of the true SNR values. These lower bounds turn out to be inversely proportional to the number of independent data records. Moreover, the CRLBs computed using our analytical expressions are similar to those computed empirically in other recent works. Finally, the derived expressions are of great value in that they allow to analyze the achievable performance and quantify the performance of SNR estimators operating on square QAM-modulated signals.

REFERENCES

[1] N. Celandroni, E. Ferro and F. Potorti, "Quality Estimation of PSK Modulated Signals," *IEEE Commun. Mag.,* pp. 50-55, July 1997.

[2] K. Balachandran, S. R. Kadaba, and S. Nanda, "Channel Quality Estimation and Rate Adaptation for Cellular Mobile Radio," *IEEE J. Select. Areas Commun.,* vol. 17, pp. 1244-1256, July 1999.

[3] P. Gao and C. Tepedelenlioglu, "SNR Estimation for Non-Constant Modulus Constellations," *IEEE Trans. Signal Processing,* vol. 53, pp. 865-870, Mar. 2005.

[4] L. Chang, F. Wang, J. Xu and Z. Wang, "Blind SNR Estimation for Arbitrary Linear Modulated Signals," *IEEE GLOBECOM,* pp. 4025-4029, 2007.

[5] N. S. Alagha, "Cramer-Rao Bounds of SNR Estimates for BPSK and QPSK Modulated Signals," *IEEE Commun. Lett.,* vol. 5, pp. 10-12, Jan. 2001.

[6] W. Gappmair, "Cramer-Rao Lower Bound for Non-Data-Aided SNR Estimation of Linear Modulation Schemes ," *IEEE Trans. Commun.,* vol. 56, pp. 689-693 , May 2008.

[7] A. Das (Nandan), "NDA SNR Estimation : CRLBs and EM Based Estimators," *accepted* for publication in *IEEE TENCON,* Nov. 2008.

[8] S. M. Kay, *Fundamentals of Statistical Signal Processing, Estimation Theory,* Upper Saddle River, NJ : Prentice Hall, 1993.